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Abstract

This paper presents a new finite volume scheme for the steady Stokes equations on a general 2D mesh. The scheme is staggered, i.e. the discrete velocities are not located at the same place as the discrete pressures. We prove the existence and the uniqueness of a discrete solution. We then prove convergence of the discrete velocities to the weak solution of the problem. Under additional regularity conditions, we prove the convergence of a penalized version of the scheme to the weak solution of the problem. Numerical experiments on problems with known analytical solutions allow to obtain the rate of convergence for both velocities and pressure.

Key words : Stokes equations, cell-centered finite volumes, unstructured mesh.

1 Introduction

We study the following problems: find an approximation of $u = (u^{(1)}, u^{(2)})^t \in H_0^1(\Omega) \times H_0^1(\Omega)$ and $p \in L^2(\Omega)$, weak solution to the generalized Stokes equations, which write:

$$\begin{array}{l} \eta u - \nu \Delta u + \nabla p = f \text{ in } \Omega, \\ \operatorname{div} u = \partial_1 u^{(1)} + \partial_2 u^{(2)} = 0 \text{ in } \Omega, \end{array} \tag{1}$$

where $\eta \geq 0$, $u^{(1)}$ and $u^{(2)}$ are the two components of the velocity, p denotes the pressure, ν the viscosity of the fluid, under the following assumptions:

$$\Omega \text{ is a polygonal open bounded connected subset of } \mathbb{R}^2, \tag{2}$$

$$\nu \in (0, +\infty), \ \ \eta \in [0, +\infty), \tag{3}$$

$$f = (f^{(1)}, f^{(1)})^t \in (L^2(\Omega))^2, \quad \text{for } i = 1, 2. \tag{4}$$

The case $\eta = 0$ yields the usual steady-state equations.

For the simplicity of this presentation, we prescribe for both problems a homogeneous Dirichlet boundary condition on the velocity $(u^{(1)}, u^{(2)})$. In all this paper, we denote by $x = (x^{(1)}, x^{(2)})$ any point of Ω and by dx the 2-dimensional Lebesgue measure $dx = dx^{(1)} dx^{(2)}$.

Definition 1.1 (Weak solution) Under hypotheses (2)-(4), let

$$E(\Omega) := \{v = (v^{(1)}, v^{(2)})^t \in (H_0^1(\Omega))^2, \operatorname{div} v = \partial_1 v^{(1)} + \partial_2 v^{(2)} = 0 \text{ a.e.}\}. \tag{5}$$

Then $u = (u^{(1)}, u^{(2)})^t$ is called a weak solution of (1) (see e.g. [24]) if and only if

$$\left\{ \begin{array}{l} u = (u^{(1)}, u^{(2)})^t \in E(\Omega), \\ \eta \displaystyle\int_\Omega u(x) \cdot v(x) dx + \nu \int_\Omega \nabla u(x) : \nabla v(x) dx = \\ \displaystyle\int_\Omega f(x) v(x) dx, \quad \forall v = (v^{(1)}, v^{(2)})^t \in E(\Omega), \end{array} \right. \tag{6}$$

where, by definition, $\nabla u : \nabla v(x) = \displaystyle\sum_{i=1,2} \nabla u^{(i)}(x) \cdot \nabla v^{(i)}(x)$.

Numerical schemes for the Stokes equations (6) (and the nonlinear Navier-Stokes equations) have been extensively studied: see [11, 20, 21, 22, 13, 12] and references therein. Among different schemes, finite element schemes and finite volume schemes are frequently used for mathematical or engineering studies. An advantage of the finite volume schemes is that the unknowns are approximated by piecewise constant functions: this makes it easy to take into account additional nonlinear phenomena or the coupling with algebraic or differential equations, for instance in the case of reactive flows; in particular, one can find in [20] the presentation of the classical finite volume scheme on rectangular meshes, which has been the basis of many industrial applications. Proofs of the convergence of the co-called “MAC scheme” [16] were performed in [19] and [23] for the Stokes equations. However, the use of rectangular grids makes an important limitation to the type of domain which can be gridded and more recently, finite volume schemes for the Navier-Stokes equations on triangular grids have been presented: see for example [14] where the vorticity formulation is used and [4] where primal variables are used with a Chorin type projection method to ensure the divergence condition. In this paper, we propose a method which uses the primitive variables and enforces the divergence condition directly, using quite general meshes such as mixed rectangular-triangular or Voronoï meshes. This finite volume scheme can be presented under the following variational form:

$$\begin{aligned}
&u = (u^{(1)}, u^{(2)})^t \in E_{\mathcal{D}}(\Omega),\\
&\eta \int_\Omega u(x) \cdot v(x) dx + \nu [u,v]_{\mathcal{D}} = \int_\Omega f(x) \cdot v(x) dx,\\
&\forall v = (v^{(1)}, v^{(2)})^t \in E_{\mathcal{D}}(\Omega).
\end{aligned} \tag{7}$$

where $E_{\mathcal{D}}(\Omega)$ is the nonconforming discrete functional space consisting of the pairs of piecewise constant functions on the cells of the mesh $\mathcal{D}$, satisfying a discrete divergence–free condition, endowed with the inner product $[\cdot,\cdot]_{\mathcal{D}}$ (see Definition 3.1). The discrete unknowns of this scheme are the discrete velocities located at some point within the discretization cells (or "control volumes" of the mesh (see Section 2) whereas the discrete divergence–free condition is imposed at the vertices of the mesh. Because of this choice, any iterative procedure will introduce discrete pressures located at the vertices of the mesh. Therefore this scheme is also a staggered scheme, in the sense that the velocities are not colocated with the pressures. It is a slight modification of a scheme which was first proposed in [6]; its convergence was proved in the linear case for an equilateral triangular mesh in [6] and after the tentative generalization of [1], an error analysis was derived in the same setting in [3]. In fact, the numerical results which we obtain here show that the modification which we use here is crucial for the scheme to perform well on any mesh (even though the original scheme and the modified scheme are identical on equilateral meshes).

This paper is organized as follows. In Section 2, we introduce the admissible finite volume meshes and we describe the discretization scheme, as well as some discrete functional tools. In Section 5, we prove the existence and uniqueness of discrete velocities solution to (7) on general structured or unstructured finite volume meshes (note that for general meshes, no inf-sup condition can be proven; this prevents from deducing the uniqueness of a discrete pressure in the whole discrete space, see also Remark 3). We then prove the convergence and an error estimate for a penalized version of the scheme. The error estimate is a power 1/4 of the mesh size and is clearly not sharp, as shown by the numerical results given in Section 9.

2 Admissible discretization of Ω

We first present the following notion of admissible discretization, which is an extension of [6].

Definition 2.1 **[Admissible discretization]** Let Ω be an open bounded polygonal subset of $\mathbb{R}^2$, and $\partial\Omega = \overline{\Omega} \setminus \Omega$ its boundary. An admissible finite volume discretization of Ω, denoted by $\mathcal{D}$, is given by $\mathcal{D} = (\mathcal{M}, \mathcal{E}, \mathcal{P}, \mathcal{V})$, where:

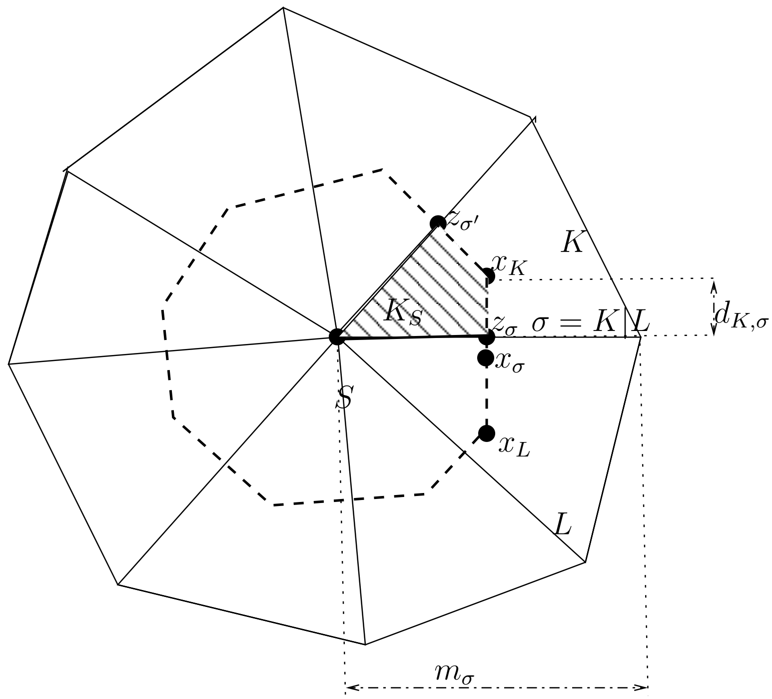


Figure 1: Example of an admissible triangular discretization

- $\mathcal{M}$ is a finite family of non empty open polygonal convex disjoint subsets of Ω (the "control volumes") such that $\overline{\Omega} = \cup_{K\in\mathcal{M}}\overline{K}$. For any $K \in \mathcal{M}$, let $\partial K = \overline{K} \setminus K$ be the boundary of K and $\mathrm{m}(K) > 0$ denote the area of K.

- $\mathcal{E}$ is a finite family of disjoint subsets of $\overline{\Omega}$ (the "edges" of the mesh), such that, for all $\sigma \in \mathcal{E}$, there exists a hyperplane E of $\mathbb{R}^2$ and $K \in \mathcal{M}$ with $\overline{\sigma} = \partial K \cap E$ and σ is a non empty open subset of E. We then denote by $m_\sigma > 0$ the 1-dimensional measure of σ. We assume that, for all $K \in \mathcal{M}$, there exists a subset $\mathcal{E}_K$ of $\mathcal{E}$ such that $\partial K = \cup_{\sigma\in\mathcal{E}_K}\overline{\sigma}$. It then results from the previous hypotheses that, for all $\sigma \in \mathcal{E}$, either $\sigma \subset \partial\Omega$ or there exists $(K, L) \in \mathcal{M}^2$ with $K \neq L$ such that $\overline{K} \cap \overline{L} = \overline{\sigma}$; we denote in the latter case $\sigma = K|L$.

- $\mathcal{P}$ is a family of points of Ω indexed by $\mathcal{M}$, denoted by $\mathcal{P} = (x_K)_{K\in\mathcal{M}}$. The coordinates of x_K are denoted by $x_K^{(i)}$, $i = 1, 2$. The family $\mathcal{P}$ is such that, for all $K \in \mathcal{M}$, $x_K \in K$. Furthermore, for all $\sigma \in \mathcal{E}$ such that there exists $(K, L) \in \mathcal{M}^2$ with $\sigma = K|L$, it is assumed that the straight line (x_K, x_L) going through x_K and x_L is orthogonal to $K|L$. For all $K \in \mathcal{M}$ and all $\sigma \in \mathcal{E}_K$, let z_σ be the orthogonal projection of x_K on σ. We suppose that $z_\sigma \in \sigma$.

- $\mathcal{V}$ is a finite family of non empty open polygonal disjoint subsets of Ω (constituting the "dual mesh" of $\mathcal{M}$), which are centered around the vertices $(x_s)_{s=1,N_V}$ in the following way (N_V is the number of vertices):

  for $1 \leq s \leq N_V$, let $\mathcal{M}_s \subset \mathcal{M}$ be the set of control volumes to which x_s is a vertex. For $K \in \mathcal{M}_s$, denote by σ and $\sigma' \in \mathcal{E}_K$ the two edges of K with vertex x_s. Define K_s as the convex hull of the four points (see Figure 1)

$$(x_s, x_K, z_\sigma, z_{\sigma'}) .$$

The dual cell around x_s, denoted by S, is then defined as (also see Figure 1):

$$\overline{S} = \cup_{K \in \mathcal{M}_s} K_s.$$

Since there is a one-to-one mapping between the set $\{1, \ldots, N_V\} \subset \mathbb{N}$ and the set $\mathcal{V}$, we shall replace all subscripts s by S when dealing with the dual mesh. Let $\mathcal{V}_K$ denote the set of vertices of a given control volume K. Note that:

$$\overline{K} = \cup_{x_s \in \mathcal{V}_K} K_s, \text{ and } K_s = \overline{K \cap S}.$$

The following notations are used. The size of the discretization is defined by:

$$h_\mathcal{D} = \sup\{\mathrm{diam}(K), K \in \mathcal{M}\}.$$

We shall measure the regularity of the mesh through the function angle($\mathcal{D}$) defined by

$$\mathrm{angle}(\mathcal{D}) = \inf\left\{|\widehat{z_\sigma x_K x_S}|,\ |\widehat{z_\sigma x_S x_K}|, K \in \mathcal{M},\ S \in \mathcal{V}_K,\ \sigma \in \mathcal{E}_K \cap \mathcal{E}_S\right\}, \qquad (8)$$

where $|\widehat{xyz}|$ designates the absolute value of the measure of the angle $\widehat{xyz}$ (note that $\widehat{z_\sigma x_K x_S} = \frac{\pi}{2} - \widehat{z_\sigma x_S x_K}$.).

For all $K \in \mathcal{M}$ and $\sigma \in \mathcal{E}_K$, we denote by $\mathbf{n}_{K,\sigma}$ the unit vector normal to σ outward to K. We denote by $d_{K,\sigma}$ the Euclidean distance between x_K and σ. We then define

$$\tau_{K,\sigma} = \frac{m_\sigma}{d_{K,\sigma}}.$$

The set of interior (resp. boundary) edges is denoted by $\mathcal{E}_{\mathrm{int}}$ (resp. $\mathcal{E}_{\mathrm{ext}}$), that is $\mathcal{E}_{\mathrm{int}} = \{\sigma \in \mathcal{E};\ \sigma \not\subset \partial\Omega\}$ (resp. $\mathcal{E}_{\mathrm{ext}} = \{\sigma \in \mathcal{E};\ \sigma \subset \partial\Omega\}$). For any $\sigma \in \mathcal{E}_{\mathrm{int}}, \sigma = K|L$ (resp. $\mathcal{E}_{\mathrm{ext}}$, $\sigma \in \mathcal{E}_K$), let x_σ be the center point of the line segment $[x_K x_L]$ (resp. $[x_K z_\sigma]$), and $x_\sigma^{(1)}$ and $x_\sigma^{(2)}$ its coordinates.

For all $K \in \mathcal{M}$ and all $S \in \mathcal{V}_K$, let σ_1 and $\sigma_2 \in \mathcal{E}_K \cap \mathcal{E}_S$ numbered such that $(x_{\sigma_1}^{(2)} - x_S^{(2)})(x_{\sigma_2}^{(1)} - x_S^{(1)}) - (x_{\sigma_2}^{(2)} - x_S^{(2)})(x_{\sigma_1}^{(1)} - x_S^{(1)}) > 0$. We then define $A_{K,S} \in \mathbb{R}^2$ by:

$$\begin{aligned} A_{K,S}^{(1)} &= x_{\sigma_1}^{(2)} - x_{\sigma_2}^{(2)} \\ A_{K,S}^{(2)} &= x_{\sigma_2}^{(1)} - x_{\sigma_1}^{(1)}. \end{aligned} \qquad (9)$$

Remark 1 (Comparison with a former finite volume scheme) Note that the former finite volume scheme which was studied in [6], [1] and [3] may be formulated in the same way, now taking for $(x_\sigma^{(1)}, x_\sigma^{(2)})$ the components of the point z_σ, which is the intersection of the line segment between x_K and x_L, and the edge σ (rather the point x_σ, middle point of x_K and x_L in the present scheme, see Figure 1). Note that for equilateral triangles, the points z_σ and x_σ coincide, and therefore the two schemes are equivalent.

Let us now state some properties which are induced by requiring some classical regularity on the mesh.

Proposition 2.2 **[Geometric properties of regular meshes]** Under hypothesis (2), Let $\mathcal{D}$ be an admissible discretization of Ω in the sense of Definition 2.1 and let $\alpha > 0$ such that $\mathrm{angle}(\mathcal{D}) \geq \alpha$. Then there exist strictly positive real numbers C_i, for $i = 1, \ldots, 5$, only depending on α, such that

$$C_1 \leq \tau_{K,\sigma}, \ \forall K \in \mathcal{M}, \ \forall \sigma \in \mathcal{E}_K, \tag{10}$$

$$\mathrm{card}(\mathcal{M}_S) \leq C_2\,, \ \forall S \in \mathcal{V}, \tag{11}$$

$$\mathrm{card}(\mathcal{V}_K) \leq C_2\,, \ \forall K \in \mathcal{M}, \tag{12}$$

$$\mathrm{diam}(K \cap S)^2 \leq C_3\, \mathrm{m}(K \cap S), \ \forall K \in \mathcal{M}, \ \forall S \in \mathcal{V}_K, \tag{13}$$

$$\mathrm{diam}(S)^2 \leq C_4\, \mathrm{m}(S), \ \forall S \in \mathcal{V}, \tag{14}$$

$$\mathrm{diam}(K)^2 \leq C_5\, \mathrm{m}(K), \ \forall K \in \mathcal{M}. \tag{15}$$

Proof The following relations hold in triangles x_K, z_σ, x_S (see Figure 1), for $K \in \mathcal{M}$, $S \in \mathcal{V}_K$ and $\sigma \in \mathcal{E}_K \cap \mathcal{E}_S$:

$$\tan|\widehat{x_S x_K z_\sigma}| = \frac{d(x_S, z_\sigma)}{d(x_K, z_\sigma)} \geq \tan\alpha, \text{ and } \tan|\widehat{x_K x_S z_\sigma}| = \frac{d(x_K, z_\sigma)}{d(x_S, z_\sigma)} \geq \tan\alpha. \tag{16}$$

Therefore we get (10) with $C_1 = 2\tan\alpha$. Using $|\widehat{x_K x_S z_\sigma}| \geq \alpha$, we get, adding the measures of the angles around x_S,

$$\mathrm{card}(\mathcal{M}_S) \leq C_2 = \frac{\pi}{\alpha},$$

and similarly, since $|\widehat{z_\sigma x_K x_S}| \geq \alpha$, we get, adding the measures of the angles around x_K,

$$\mathrm{card}(\mathcal{E}_K) = \mathrm{card}(\mathcal{V}_K) \leq C_2 = \frac{\pi}{\alpha}.$$

We thus prove (11) and (12).

Let us now remark that

$$\begin{aligned} \mathrm{meas}(K \cap S) &= \frac{1}{2} d(x_S, x_K)^2 (\sin \widehat{z_\sigma x_S x_K} \sin \widehat{x_S x_K z_\sigma} + \sin \widehat{z_{\sigma'} x_S x_K} \sin \widehat{x_S x_K z_{\sigma'}}) \\ &\geq (\sin\alpha)^2 d(x_S, x_K)^2, \end{aligned}$$

where σ and σ' are the two edges of K intersecting at S. This proves (13).

Let us then remark that

$$\mathrm{m}(S) = \sum_{K \in \mathcal{M}_S} \mathrm{m}(K \cap S) \geq (\sin\alpha)^2 \sum_{K \in \mathcal{M}_S} d(x_S, x_K)^2,$$

so that

$$\mathrm{m}(S) \geq (\sin\alpha)^2 \max_{K \in \mathcal{M}_S} d(x_S, x_K)^2 \geq (\sin\alpha)^2 \left(\frac{\mathrm{diam}(S)}{2}\right)^2,$$

which proves (14).

Similarly,

$$\mathrm{m}(K) = \sum_{S\in\mathcal{V}_K} \mathrm{m}(K\cap S) \geq (\sin\alpha)^2 \sum_{S\in\mathcal{V}_K} d(x_S,x_K)^2 \geq (\sin\alpha)^2 \left(\frac{\mathrm{diam}(K)}{2}\right)^2,$$

which proves (15).

3 Discrete functional properties

Definition 3.1 Let Ω be an open bounded polygonal subset of $\mathbb{R}^N$, with $N \in \mathbb{N}_*$. Let $\mathcal{D} = (\mathcal{M},\mathcal{E},\mathcal{P},\mathcal{V})$ be an admissible finite volume discretization of Ω in the sense of Definition 2.1. We denote by $H_{\mathcal{D}}(\Omega) \subset L^2(\Omega)$ the space of functions which are piecewise constant on each control volume $K \in \mathcal{M}$. For all $w \in H_{\mathcal{D}}(\Omega)$ and for all $K \in \mathcal{M}$, we denote by w_K the constant value of w in K. For $\sigma = K|L \in \mathcal{E}_{\mathrm{int}}$, we define w_σ to be the linear interpolation between w_K and w_L if $\sigma = K|L$, and in this case the following relationship (conservation of the local fluxes) is satisfied:

$$\tau_{K,\sigma}(w_\sigma - w_K) + \tau_{L,\sigma}(w_\sigma - w_L) = 0,\ \forall \sigma \in \mathcal{E}_{\mathrm{int}}, \sigma = K|L. \tag{17}$$

and for $\sigma \in \mathcal{E}_{\mathrm{ext}}$, we set

$$w_\sigma = 0,\ \forall \sigma \in \mathcal{E}_{\mathrm{ext}}. \tag{18}$$

and

We denote by $L_{\mathcal{D}}(\Omega)$ the space of functions which are piecewise constant on the domains S, for all $S \in \mathcal{V}$. We then define the discrete divergence operator $\mathrm{div}_{\mathcal{D}}\ : (H_{\mathcal{D}}(\Omega))^2 \to L_{\mathcal{D}}(\Omega)$, by:

$$\mathrm{div}_{\mathcal{D}}(u)(x) = \frac{1}{\mathrm{m}(S)} \sum_{K\in\mathcal{M}_S} A_{K,S}\cdot\ u_K,\quad \text{for a.e. } x \in S, \forall S \in \mathcal{V}.$$

We then set $E_{\mathcal{D}}(\Omega) = \{u \in (H_{\mathcal{D}}(\Omega))^2, \mathrm{div}_{\mathcal{D}}(u) = 0\}$. For v and w elements of $H_{\mathcal{D}}(\Omega))$, we denote by

$$[v,w]_{\mathcal{D}} = \sum_{K\in\mathcal{M}} \sum_{\sigma\in\mathcal{E}_K} \tau_{K,\sigma}(v_\sigma - v_K)(w_\sigma - w_K), \tag{19}$$

the inner product on $H_{\mathcal{D}}(\Omega)$. Remark that thanks to (17), one has:

$$[v,w]_{\mathcal{D}} = \sum_{\sigma\in\mathcal{E}_{\mathrm{int}},\sigma=K|L} \tau_\sigma(v_K - v_L)(w_K - w_L) + \sum_{\sigma\in\mathcal{E}_{\mathrm{ext}}} \tau_\sigma v_{K_\sigma} w_{K_\sigma},$$

where K_σ denotes the control volume to which σ is an edge. For simplicity, we shall also denote by $[,]$ the inner product on $(H_{\mathcal{D}}(\Omega))^2$, that is:

$$[v,w]_{\mathcal{D}} = \sum_{K\in\mathcal{M}} \sum_{\sigma\in\mathcal{E}_K} \sum_{i=1,2} \tau_{K,\sigma}(v_\sigma^{(i)} - v_K^{(i)})(w_\sigma^{(i)} - w_K^{(i)}),\ \forall (v,w) \in (H_{\mathcal{D}}(\Omega))^2.$$

We define a norm in $H_\mathcal{D}(\Omega)$ (resp. $(H_\mathcal{D}(\Omega))^2$), thanks to the discrete Poincaré inequality (20) given below, by

$$\|w\|_\mathcal{D} = ([w,w]_\mathcal{D})^{1/2}.$$

The discrete Poincaré inequality (see [6]) reads as:

$$\|w\|_{L^2(\Omega)} \leq \mathrm{diam}(\Omega)\|w\|_\mathcal{D},\ \forall w \in H_\mathcal{D}(\Omega). \tag{20}$$

4 The finite volume scheme

Under hypotheses (2)-(4), let $\mathcal{D}$ be an admissible discretization of Ω in the sense of Definition 2.1. The finite volume approximate problem to the Stokes problem (6) is defined by the following set of equations:

$$\left\{\begin{array}{l} u =\in E_\mathcal{D}(\Omega), \\ \eta \displaystyle\int_\Omega u(x)\cdot (x)dx + \nu[u,v]_\mathcal{D} = \\ \int_\Omega f(x)\cdot v(x)dx,\ \forall v \in E_\mathcal{D}(\Omega). \end{array}\right. \tag{21}$$

The existence and the uniqueness of a solution to (21) will be stated in Proposition 5.3. The numerical approximation of a solution to (21) is usually obtained by introducing the pressure, which may be seen as a Lagrange multiplier to the divergence–free constraint. There are several methods to solve the velocities-pressure system, which are more or less efficient depending on the data of the problem (see e.g. [11, 12]). Among them, we select a penalized scheme, since it allows the derivation of a mathematical proof of convergence. This penalized scheme, which we now state, was introduced and analysed in [11] in the framework of a finite element discretization. For any real value $\lambda > 0$, we look for u such that

$$\begin{array}{l} u = (u^{(1)}, u^{(2)})^t \in (H_\mathcal{D}(\Omega))^2, \\ \eta \int_\Omega u(x)\cdot v(x)dx + \nu[u,v]_\mathcal{D} + \dfrac{1}{\lambda\ h_\mathcal{D}} \displaystyle\int_\Omega \mathrm{div}_\mathcal{D}(u)\mathrm{div}_\mathcal{D}(v) = \\ \displaystyle\int_\Omega f(x)\cdot v(x)dx,\ \forall v \in (H_\mathcal{D}(\Omega))^2, \end{array} \tag{22}$$

or, introducing a discrete pressure, we look for (u,p) such that

$$\begin{array}{l} (u,p) \in (H_\mathcal{D}(\Omega))^2 \times L_\mathcal{D}(\Omega), \\ \eta \int_\Omega u(x)\cdot v(x)dx + \nu[u,v]_\mathcal{D} - \displaystyle\int_\Omega p(x)\mathrm{div}_\mathcal{D}(v)(x)dx = \\ \displaystyle\int_\Omega f\cdot (x)v(x)dx,\ \forall v \in (H_\mathcal{D}(\Omega))^2, \\ \mathrm{div}_\mathcal{D}(u) = -\lambda\ h_\mathcal{D}\ p. \end{array} \tag{23}$$

System (23) is equivalent to finding the values $(u_K)_{K\in\mathcal{M}}$, $(u_\sigma)_{\sigma\in\mathcal{E}}$, $(v_K)_{K\in\mathcal{M}}$, $(v_\sigma)_{\sigma\in\mathcal{E}}$ and $(p_S)_{S\in\mathcal{V}}$ solution of the following system of equations:

$$\left\{\begin{array}{l}\eta\ \mathrm{m}(K)u_K-\nu\displaystyle\sum_{\sigma\in\mathcal{E}_K}\tau_{K,\sigma}(u_\sigma-u_K)-\sum_{S\in\mathcal{V}_K}A_{K,S}\ p_S=\int_K f(x)dx,\forall K\in\mathcal{M},\\[2ex] \tau_{K,\sigma}(u_\sigma-u_K)+\tau_{L,\sigma}(u_\sigma-u_L)=0,\ \forall\sigma\in\mathcal{E}_{\mathrm{int}}\text{ with }\sigma=K|L,\\[2ex] u_\sigma=0,\ \forall\sigma\in\mathcal{E}_{\mathrm{ext}},\end{array}\right. \tag{24}$$

$$\sum_{K\in\mathcal{M}_S}\sum_{i=1,2}A_{K,S}\cdot u_K=-\lambda\ h_{\mathcal{D}}\mathrm{m}(S)p_S,\ \forall S\in\mathcal{V}. \tag{25}$$

Remark 2

1. In System (24), the values $(u_\sigma^{(i)})_{\sigma\in\mathcal{E}}$, $i=1,2$, can easily be eliminated using the last two equations of (24), thus yielding a linear system, the unknowns of which are the values $(u_K^{(i)})_{K\in\mathcal{M}}$, $i=1,2$, and $(p_S)_{S\in\mathcal{V}}$. This is done in practice when implementing this scheme.

2. The above scheme (23) coincides, in the case of an equilateral triangular mesh, with that of [6] for which a convergence proof was given in [6] and an error estimate in [3].

The existence of a solution to (23) will be proven below.

5 Existence and uniqueness of the discrete solution

In order to prove existence and uniqueness of a solution to the finite volume scheme, we start with the following estimate:

PROPOSITION 5.1 **[Discrete H^1 estimate on velocities, weak L^2 inequality on pressures]** Under hypotheses (2)-(4), let $\mathcal{D}$ be an admissible discretization of Ω in the sense of Definition 2.1. Let $\lambda\in(0,+\infty)$ be given. Let $(u,p)\in(H_{\mathcal{D}}(\Omega))^2\times L_{\mathcal{D}}(\Omega)$ be a solution to (23). Then the following inequalities hold:

$$\nu\|u\|_{\mathcal{D}}\le\mathrm{diam}(\Omega)\|f\|_{(L^2(\Omega))^2}, \tag{26}$$

and

$$(\nu\ \lambda\ h_{\mathcal{D}})^{1/2}\ \|p\|_{L^2(\Omega)}\le\mathrm{diam}(\Omega)\|f\|_{(L^2(\Omega))^2}. \tag{27}$$

Proof We apply (23) setting $v=u$. We get

$$\eta\int_\Omega u(x)\cdot v(x)dx+\nu\|u\|_{\mathcal{D}}^2-\int_\Omega p(x)\mathrm{div}_{\mathcal{D}}(u)(x)dx=\int_\Omega f(x)\cdot v(x)dx.$$

Since $\eta\ge 0$, the second equation of (23) and Young's inequality yield that:

$$\eta\int_\Omega u(x)\cdot v(x)dx+\nu\|u\|_{\mathcal{D}}^2+\lambda\ h\ \|p\|_{L^2(\Omega)}^2\le\frac{\mathrm{diam}(\Omega)^2}{2\nu}\|f\|_{(L^2(\Omega))^2}^2+\frac{\nu}{2\mathrm{diam}(\Omega)^2}\|u\|_{(L^2(\Omega))^2}^2.$$

Using the Poincaré inequality (20) gives

$$\nu\|u\|_{\mathcal{D}}^2 + \lambda\ h\ \|p\|_{L^2(\Omega)}^2 \le \frac{\mathrm{diam}(\Omega)^2}{2\nu}\|f\|_{(L^2(\Omega))^2}^2 + \frac{\nu}{2}\|u\|_{\mathcal{D}}^2,$$

which leads to (26). Then reporting (26) in the above inequality yields (27).

We can now state the existence and the uniqueness of a discrete solution to (23). We first give the result for the case $\lambda > 0$.

Corollary 5.2 **[Existence and uniqueness of a solution to the penalized finite volume scheme]** Under hypotheses (2)-(4), let $\mathcal{D}$ be an admissible discretization of Ω in the sense of Definition 2.1. Let $\lambda \in (0, +\infty)$ be given. Then there exists a unique solution to (23).

Proof System (23) is a linear system. Assume that $f = 0$. From Proposition 5.1, we get that $u = 0$ and $p = 0$. This proves that the linear system (23) is invertible.

We now state the existence and uniqueness result for the case $\lambda = 0$.

Proposition 5.3 **[Existence and uniqueness of a solution to the finite volume scheme]** Under hypotheses (2)-(4), let $\mathcal{D}$ be an admissible discretization of Ω in the sense of Definition 2.1. Then there exists a unique solution u to (21).

Proof The uniqueness result is an immediate consequence of (20), setting $v = u$ in (21).

Let us now prove the existence. Let $(\lambda^{(n)})_{n\in\mathbb{N}}$ be a sequence of strictly positive real values such that $\lambda^{(n)} \to 0$ as $n \to \infty$. For all $n \in \mathbb{N}$, let $(u^{(n)}, p^{(n)}) \in (H_{\mathcal{D}}(\Omega))^2 \times L_{\mathcal{D}}(\Omega)$ be the solution to (23) with $\lambda = \lambda^{(n)}$. Using (26) and (20), we get that the sequences $(u_K^{(n)})_{n\in\mathbb{N}}$ is bounded, for all $K \in \mathcal{M}$. We can therefore extract a subsequence of $(\lambda^{(n)})_{n\in\mathbb{N}}$, again denoted $(\lambda^{(n)})_{n\in\mathbb{N}}$, such that the sequence $(u_K^{(n)})_{n\in\mathbb{N}}$ converge, for all $K \in \mathcal{M}$. Let us denote $u \in (H_{\mathcal{D}}(\Omega))^2$ the piecewise constant function whose value on a cell K is the limit of the sequence $(u_K^{(n)})_{n\in\mathbb{N}}$; it is easily seen that u is also the limit in $(L^2(\Omega))^2$ of the sequence $(u^{(n)})_{n\in\mathbb{N}}$. Moreover, for all $v \in E_{\mathcal{D}}(\Omega)$, the following equality holds:

$$\eta\int_\Omega u^{(n)}(x)\cdot v(x)dx + \nu[u^{(n)}, v]_{\mathcal{D}} = \int_\Omega f(x)\cdot v(x)dx, \text{ for all } n \in \mathbb{N},$$

we only have to verify that $u \in E_{\mathcal{D}}(\Omega)$. From (23), one has:

$$\int_\Omega \Big(\mathrm{div}_{\mathcal{D}}(u^{(n)})(x)\Big)^2 dx = (\lambda^{(n)}\ h)^2 \sum_{S\in\mathcal{V}} \mathrm{m}(S) p_S^2.$$

Thanks to (27), we get

$$\|\mathrm{div}_{\mathcal{D}}(u^{(n)})\|_{L^2(\Omega)}^2 \le \lambda^{(n)}\ h_{\mathcal{D}} \frac{\mathrm{diam}(\Omega)^2}{\nu}\|f\|_{(L^2(\Omega))^2}^2. \tag{28}$$

Now

$$\|\mathrm{div}_{\mathcal{D}}(u^{(n)})\|_{L^2(\Omega)}^2 = \sum_{S\in\mathcal{V}} \frac{1}{\mathrm{m}(S)} \sum_{i=1,2} \sum_{K\in\mathcal{M}_S} (A_{K,S}^{(i)}\ u_K^{(n,i)})^2,$$

and therefore one may pass to the limit as $n \to \infty$ in the Equation (28). This yields $\mathrm{div}_{\mathcal{D}}(u) = 0$ which implies $u \in E_{\mathcal{D}}(\Omega)$.

Remark 3 From the estimate on the solutions u of the penalized scheme (23), which does not depend on λ, we can also get the existence and uniqueness of $(u,p) \in (H_{\mathcal{D}}(\Omega))^2 \times (Ker(\nabla_{\mathcal{D}}))^{\perp}$ solution of (23) with $\lambda = 0$, where $\nabla_{\mathcal{D}}$ denotes the operator from $L_{\mathcal{D}}(\Omega)$ to $(H_{\mathcal{D}}(\Omega))^2$ which is the tranposed of $\mathrm{div}_{\mathcal{D}}$.

6 Consistency properties

We now study consistency properties of the discrete operators which will be needed in the proof of convergence. Let us first recall that the approximation of the Laplace operator by the above finite volume scheme is known to be of order 1: see e.g. [10], [6]. Let us then turn to the other operators.

Proposition 6.1 **[Consistency of the discrete divergence]** Under hypothesis (2), let $\mathcal{D}$ be an admissible discretization of Ω in the sense of Definition 2.1, and let $\alpha > 0$ such that $\mathrm{angle}(\mathcal{D}) \geq \alpha$.

Then, there exists C_6 , only depending on α, such that, for all $\bar{u} \in H_0^1(\Omega) \cap H^2(\Omega)$, and for any $S \in \mathcal{V}$,

$$\left| \sum_{K \in \mathcal{M}_S} A_{K,S}^{(k)} \, \bar{u}^{(k)}(x_K) - \int_{\partial S} \bar{u}^{(k)}(x) \mathbf{e_k} \cdot \mathbf{n}(x) d\gamma(x) \right| \leq C_6 \, \mathrm{m}(S) |\bar{u}^{(k)}|_{H^2(S)}, \; k = 1, 2, \tag{29}$$

where

$$|\bar{u}^{(k)}|_{H^2(S)} = \left(\sum_{i,j=1}^{2} \int_S \left(\frac{\partial^2 \bar{u}^{(k)}}{\partial x_i \partial x_j}(x) \right)^2 dx \right)^{1/2}, \; k = 1, 2.$$

Remark 4 Remark that

$$\int_{\partial S} \bar{u}^{(k)}(x) \mathbf{e_k} \cdot \mathbf{n}(x) d\gamma(x) = \int_S \partial_k \overline{u},$$

so that one may see the term

$$\frac{1}{\mathrm{m}(S)} \sum_{K \in \mathcal{M}_S} A_{K,S}^{(k)} \, \bar{u}^{(k)}(x_K)$$

as a consistent discretization of $\partial_k \overline{u}$ over the cell S, for $k = 1, 2$.

Proof Under the assumptions of Proposition 6.1, let $S \in \mathcal{V}$, we wish to show that $|T_S^{(k)}| \leq C_6 \, \mathrm{m}(S) |\bar{u}^{(k)}|_{H^2(S)}$, where:

$$T_S^{(k)} = \sum_{K \in \mathcal{M}_S} A_{K,S}^{(k)} \, \bar{u}^{(k)}(x_K) - \int_{\partial S} \bar{u}^{(k)}(x) \mathbf{e_k} \cdot \mathbf{n}(x) d\gamma(x).$$

Let us first consider the case where S has no edge on the boundary of Ω. Then ∂S can be given by the sequence of segments $[x_{K_1} x_{K_2}]$, $[x_{K_2} x_{K_3}] \ldots [x_{K_{m-1}} x_{K_m}]$, $[x_{K_m} x_{K_1}]$, where $x_{K_1}, \ldots, x_{K_m}$ are the centers of the control volumes $\{K_1, K_2, \ldots, K_m\} = \mathcal{M}_S$, taken in the trigonometric order around S. We then denote $i^+ = i+1$ if $i = 1, 2, \ldots, m-1$ and $m^+ = 1$, and $i^- = i-1$ if $i = 2, \ldots, m$ and $1^- = m$. Then the term $T_S^{(k)}$ can be rewritten as

$$T_S^{(k)} = \sum_{i=1}^{m} A_{K_i,S}^{(k)}\ \bar{u}^{(k)}(x_{K_i}) - \int_{[x_{K_i} x_{K_{i^+}}]} \bar{u}^{(k)}(x)\mathbf{e_k}\cdot\mathbf{n}(x)d\gamma(x).$$

Using the definition of $A_{K,S}^{(k)}$ given in Definition 2.1, we get that

$$A_{K_i,S}^{(k)} = \frac{(-1)^j}{2}\left((x_{K_{i^+}}^{(j)} + x_{K_i}^{(j)}) - (x_{K_i}^{(j)} + x_{K_{i^-}}^{(j)})\right) = \frac{(-1)^j}{2}\left((x_{K_{i^+}}^{(j)} - x_{K_i}^{(j)}) + (x_{K_i}^{(j)} - x_{K_{i^-}}^{(j)})\right),$$
with $j = 2$ if $k = 1$, $j = 1$ if $k = 2$.

This yields $T_S^{(k)} = \sum_{i=1}^{m}(x_{K_{i^+}}^{(j)} - x_{K_i}^{(j)})\mathcal{L}_S^{(i)}(\bar{u})$ with

$$\mathcal{L}_S^{(i)}(\bar{u}^{(k)}) = \frac{1}{2}(\bar{u}^{(k)}(x_{K_i}) + \bar{u}^{(k)}(x_{K_{i^+}}) - \frac{1}{d_{K_i|K_{i^+}}}\int_{[x_{K_i} x_{K_{i^+}}]} \bar{u}^{(k)}(x)d\gamma(x)\ , \qquad (30)$$

since $\mathbf{e_k}\cdot\mathbf{n}(x) = (x_{K_{i^+}}^{(j)} - x_{K_i}^{(j)})/d_{K_i|K_{i^+}}$ for all $x \in [x_{K_i} x_{K_{i^+}}]$, for $i = 1, 2, \ldots, m$ (See Figure 2).

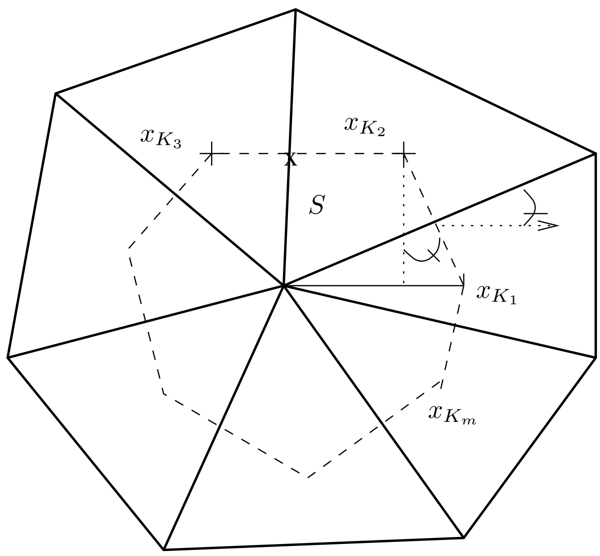


Figure 2: Consistency of the discrete divergence operator

We denote by $S^{(i)}$ the triangle $x_S, x_{K_i}, x_{K_{i^+}}$ and we remark that the operator $\mathcal{L}_S^{(i)} : H^2(S^{(i)}) \to \mathbb{R}$ defined by (30) is a continuous linear form. Let $\hat{V}$ be a given equilateral triangle $x_{S_1}, x_{S_2}, x_{S_3}$ with diameter 1, and let Ψ be the affine mapping from $S^{(i)}$ to $\hat{V}$ such that $\Psi(x_S) = x_{S_1}$, $\Psi(x_{K_i}) = x_{S_2}$ and $\Psi(x_{K_{i^+}}) = x_{S_3}$. Let

$\hat{\mathcal{L}} \in (H^2(\hat{V}))'$ be defined, for all $\hat{u} \in H^2(\hat{V})$, by $\hat{\mathcal{L}}(\hat{u}) = \mathcal{L}_S^{(i)}(\hat{u} \circ \Psi)$. Performing the change of variable $x = \Psi^{-1}(\hat{x})$, we get

$$\hat{\mathcal{L}}(\hat{u}) = \frac{1}{2}(\hat{u}(x_{S_2}) + \hat{u}(x_{S_3})) - \frac{1}{|x_{S_2} - x_{S_3}|} \int_{[x_{S_2} x_{S_3}]} \hat{u}(x) d\gamma(x).$$

Since the linear form $\hat{\mathcal{L}}$ vanishes on the set of polynomials of degree 1, we get from the Bramble-Hilbert Lemma the existence of $C_7 > 0$ (which does not depend on anything, this is to be noted since it is quite rare...), such that

$$\left|\hat{\mathcal{L}}(\hat{u})\right| \le C_7 \, |\hat{u}|_{H^2(\hat{V})}.$$

Defining, for $\bar{u} \in H^2(S^{(i)})$, the function $\hat{u}$ by $\hat{u} = \bar{u} \circ \Psi^{-1}$, we get $\mathcal{L}_S^{(i)}(\bar{u}) = \hat{\mathcal{L}}(\hat{u})$ and therefore $\left|\mathcal{L}_S^{(i)}(\bar{u})\right| \le C_7 \, |\hat{u}|_{H^2(\hat{V})}$. Let us recall, for the sake of completeness, the classical relationships between a reference element and the current element (see [18] for example):

$$\sum_{i,j=1}^{2} \left(\frac{\partial^2 \hat{u}}{\partial \hat{x}_i \partial \hat{x}_j}(\hat{x}) \right)^2 \le \|\Psi^{-1}\|^4 \sum_{i,j=1}^{2} \left(\frac{\partial^2 \bar{u}}{\partial x_i \partial x_j}(\Psi^{-1}(\hat{x})) \right)^2,$$

and $\|\Psi^{-1}\| \le \mathrm{diam}(S^{(i)})/\hat{\rho}$, where $\hat{\rho}$ is the diameter of the inscribed circle in $\hat{V}$. We thus get

$$|\hat{u}|^2_{H^2(\hat{V})} \le \frac{\mathrm{diam}(S^{(i)})^4}{\hat{\rho}^4} \frac{\mathrm{m}(\hat{V})}{\mathrm{m}(S^{(i)})} |\bar{u}|^2_{H^2(S^{(i)})}.$$

Applying (13) of Proposition 2.2, this yields $|\hat{u}|_{H^2(\hat{V})} \le C_8 \, \mathrm{diam}(S^{(i)}) |\bar{u}|_{H^2(S^{(i)})}$, where C_8 only depends on α, and therefore

$$\left|\mathcal{L}_S^{(i)}(\bar{u})\right| \le C_9 \, \mathrm{diam}(S^{(i)}) |\bar{u}|_{H^2(S^{(i)})},$$

where the non negative real C_9 only depends on α. Thanks to the Cauchy-Schwarz inequality, to the inequality $|x^{(j)}_{K_{i+}} - x^{(j)}_{K_i}| \le \mathrm{diam}(S)$ and applying (11) and (14) of Proposition 2.2, one obtains:

$$(T_S^{(k)})^2 \le {C_9}^2 C_2 \, {C_4}^2 \mathrm{m}(S)^2 |\bar{u}^{(k)}|^2_{H^2(S)}.$$

We thus obtain (29) in that case.

Let us now suppose that $\partial S \cap \partial \Omega$ is non empty. Then ∂S can be given by a part of $\partial\Omega$, the segment $[z_{\sigma_0}, x_{K_1}]$, the sequence of segments$[x_{K_1} x_{K_2}], [x_{K_2} x_{K_3}] \ldots, [x_{K_{m-1}} x_{K_m}], [x_{K_m} x_{K_1}]$, and the segment $[x_{K_m} z_{\sigma_1}]$, where again we denote by x_{K_i}, $i = 1, \ldots, m$ the centers of the control volumes $\{K_1, K_2, \ldots, K_m\} = \mathcal{M}_S$, still taken in the trigonometric order around S. We then have:

$$A^{(k)}_{K_i,S} = \frac{(-1)^j}{2} \left((x^{(j)}_{K_2} + x^{(j)}_{K_1}) - (x^{(j)}_{K_1} + x^{(j)}_{\sigma_0}) \right) = \frac{(-1)^j}{2} \left((x^{(j)}_{K_2} - x^{(j)}_{K_1}) + (x^{(j)}_{K_1} - x^{(j)}_{\sigma_0}) \right),$$

with $j = 2$ if $k = 1$, $j = 1$ if $k = 2$.

which yields $T_S^{(k)} = (x_{K_1}^{(j)} - x_{\sigma_0}^{(j)})T_S^{(k,0)} + \sum_{i=1}^{m-1}(x_{K_{i+}}^{(j)} - x_{K_i}^{(j)})T_S^{(k,i)} + (x_{\sigma_1}^{(j)} - x_{K_m}^{(j)})T_S^{(k,m)}$ with $T_S^{(k,i)}$ defined as above for $i = 1, \ldots, m-1$ and

$$T_S^{(k,0)} = \frac{1}{2}\bar{u}^{(k)}(x_{K_1}) - \frac{1}{d_{K_1,\sigma_0}}\int_{[z_{\sigma_0}x_{K_1}]}\bar{u}^{(k)}(x)d\gamma(x) \ ,$$

and

$$T_S^{(k,m)} = \frac{1}{2}\bar{u}^{(k)}(x_{K_m}) - \frac{1}{d_{K_m,\sigma_1}}\int_{[z_{\sigma_1}x_{K_m}]}\bar{u}(x)d\gamma(x) \ .$$

Since $\bar{u}^{(k)}$ vanishes in z_{σ_0} and z_{σ_1}, the application of the Bramble-Hilbert Lemma is again possible. The conclusion follows.

Proposition 6.2 **[Interpolation error for the pressure]** Under hypothesis (2), let $\mathcal{D}$ be an admissible discretization of Ω in the sense of Definition 2.1, and let $\alpha > 0$ such that $\text{angle}(\mathcal{D}) \geq \alpha$. Then, there exists C_{10}, only depending on α, such that, for all $\overline{p} \in H^1(\Omega)$, denoting by $\overline{p}^{\mathcal{D}} \in L_{\mathcal{D}}(\Omega)$ the function defined by

$$\overline{p}_S^{\mathcal{D}} = \frac{1}{\mathrm{m}(S)}\int_S \overline{p}(x)dx, \text{ for all } S \in \mathcal{V},$$

$$\left|\int_{[x_\sigma x_S]}(\overline{p} - \overline{p}^{\mathcal{D}})(x)\ d\gamma(x)\right| \leq C_{10}\,\mathrm{diam}(S)|\overline{p}|_{H^1(S)}, \tag{31}$$

where

$$|\overline{p}|_{H^1(S)} = \left(\sum_{i=1}^{2}\int_S\left(\frac{\partial\overline{p}}{\partial x_i}(x)\right)^2 dx\right)^{1/2}.$$

Proof The proof results from the Bramble-Hilbert Lemma and the fact that $\int_{[x_\sigma x_S]}(q - q^{\mathcal{D}})(x)\ d\gamma(x) = 0$, for any $q \in L_{\mathcal{D}}(\Omega)$.

7 Convergence of the finite volume scheme.

We have the following result, which states the convergence of the scheme (23), the parameter $\lambda > 0$ being fixed.

Proposition 7.1 **[Convergence of the penalized finite volume scheme in the linear case]** Under hypotheses (2)-(4), let $\lambda \in (0, +\infty)$ be given and let $(\mathcal{D}^{(n)})_{n\in\mathbb{N}}$ be a sequence of admissible discretizations of Ω in the sense of Definition 2.1, such that $\lim_{n\to\infty}\text{size}(\mathcal{D}^{(n)}) = 0$ and such that there exists $\alpha > 0$ with $\text{angle}(\mathcal{D}^{(n)}) \geq \alpha$, for all $n \in \mathbb{N}$.

Let $(u^{(n)}, p^{(n)}) \in (H_{\mathcal{D}^{(n)}}(\Omega))^2 \times L_{\mathcal{D}^{(n)}}(\Omega)$ be the solution to (23). Then the sequence $(u^{(n)})_{n\in\mathbb{N}}$ converges in $(L^2(\Omega))^2$ to u, weak solution of the Stokes problem in the sense of (6).

Proof Using (26), we obtain (see [9], [6]) an estimate on the translates of $u^{(n)}$: for all $n \in \mathbb{N}$, there exists $C_{11} > 0$, only depending on Ω, ν, f and g such that

$$\int_\Omega (u^{(n,k)}(x+\chi) - u^{(n,k)}(x))^2 dx \leq C_{11}\, |\chi|(|\chi| + 4\text{size}(\mathcal{D}^{(n)})), \text{ for } k = 1,2, \ \forall \chi \in \mathbb{R}^2, \tag{32}$$

where $u^{(n,k)}$ denotes the k-th component of $u^{(n)}$. We may then apply Kolmogorov's theorem, and obtain the existence of a subsequence of $(\mathcal{D}^{(n)})_{n\in\mathbb{N}}$ and of $u \in H_0^1(\Omega)^2$ such that $(u^{(n)})_{n\in\mathbb{N}}$ converges to u in $L^2(\Omega)^2$. It suffices now to prove that u satisfies (6) to conclude, thanks to the uniqueness of this solution. Let $\varphi \in (C_c^\infty(\Omega))^2$ such that $\text{div}(\varphi) = 0$. Let $n \in \mathbb{N}$ such that $\mathcal{D}^{(n)}$ belongs to the above extracted subsequence and let $(u^{(n)}, p^{(n)})$ be the solution to (23) with $\mathcal{D} = \mathcal{D}^{(n)}$. We suppose that n is large enough and thus size$(\mathcal{D}^{(n)})$ is small enough so that, for all $K \in \mathcal{M}$, if $K\cap$ support$(\varphi) \neq \emptyset$, then $\partial K \cap \partial\Omega = \emptyset$. Let us denote by $\varphi^{(n)}$ the element of $(H_\mathcal{D}(\Omega))^2$ with the constant value $\varphi(x_K)$ in K, for all $K \in \mathcal{M}$. Let us take $v = \varphi^{(n)}$ in (23). The convergence of the discrete Laplace operator was shown in [9] or [6]:

$$\lim_{n\to\infty} [u^{(n)}, \varphi^{(n)}]_{\mathcal{D}^{(n)}} = \int_O \nabla u(x) : \nabla\varphi(x) dx.$$

Moreover, it is clear that :

$$\lim_{n\to\infty} \int_\Omega f(x)\varphi^{(n)}(x)dx = \int_\Omega f(x)\cdot\varphi(x)dx,$$

and

$$\lim_{n\to\infty} \eta \int_\Omega u^{(n)}(x)\cdot\varphi^{(n)}(x)dx = \eta\int_\Omega u(x)\cdot\varphi(x)dx.$$

There now remains to prove that:

$$\lim_{n\to\infty} \int_\Omega p^{(n)}(x)\text{div}_{\mathcal{D}^{(n)}}(\varphi^{(n)})(x)dx = 0. \tag{33}$$

Let us write

$$\int_\Omega p^{(n)}(x)\text{div}_{\mathcal{D}^{(n)}}(\varphi^{(n)})(x)dx = \sum_{i=1,2} T_1^{(n,i)},$$

with

$$T_1^{(n,i)} = \sum_{i=1,2}\sum_{S\in\mathcal{V}^{(n)}}\sum_{K\in\mathcal{M}_S} p_S^{(n)} A_{K,S}^{(i)}\ \varphi^{(i)}(x_K), \text{ for } i = 1,2.$$

We define

$$\tilde{T}_1^{(n,i)} = \int_\Omega p^{(n)}(x)\partial_i\varphi^{(i)}(x)dx, \text{ for } i = 1,2,$$

Thanks to (27), we may apply Proposition 7.2 given below with $\theta = \frac{1}{2}$. We thus get that

$$\lim_{n\to\infty} (T_1^{(n,i)} - \tilde{T}_1^{(n,i)}) = 0, \text{ for } i = 1,2.$$

Since $\text{div}(\varphi) = 0$, we get $\displaystyle\sum_{i=1,2} T_1^{(n,i)} = 0$, which yields (33).

The last step is to prove that $\mathrm{div}(u) = 0$ a.e. in Ω. Let $\varphi \in C_c^\infty(\Omega)$. For all $n \in \mathbb{N}$, let $\varphi^{(n)} \in H_{\mathcal{D}^{(n)}}(\Omega)$ be the function defined by the value $\varphi(x_K)$ in K, for all $K \in \mathcal{M}^{(n)}$, and let $\tilde{\varphi}^{(n)} \in L_{\mathcal{D}^{(n)}}(\Omega)$ be the function defined by the value $\varphi(x_S)$ in S, for all $S \in \mathcal{V}^{(n)}$. We multiply the second equation of (23) by the function $\tilde{\varphi}^{(n)}$ and sum over $S \in \mathcal{V}^{(n)}$. We get $T_2^{(n)} = -T_3^{(n)}$, where

$$T_2^{(n)} = \sum_{S\in\mathcal{V}^{(n)}} \varphi(x_S) \sum_{K\in\mathcal{M}_S} \sum_{i=1,2} A_{K,S}^{(i)} \, u_K^{(n,i)}.$$

and

$$T_3^{(n)} = \lambda \ \mathrm{size}(\mathcal{D}^{(n)}) \int_\Omega p^{(n)}(x)\tilde{\varphi}^{(n)}(x)dx.$$

Thanks to the regularity of φ and to (11) of Proposition 2.2, there exists $C_{12} > 0$, which only depends on φ and α, such that:

$$\sum_{S\in\mathcal{V}^{(n)}} \sum_{K\in\mathcal{M}_S} (\tilde{\varphi}_S^{(n)} - \varphi_K^{(n)})^2 \le C_{12}\,,$$

we may therefore apply Proposition (7.3) given below to obtain:

$$\lim_{n\to\infty} \sum_{S\in\mathcal{V}^{(n)}} \sum_{K\in\mathcal{M}_S} u_K^{(n,i)} A_{K,S}^{(i)} \tilde{\varphi}_S^{(n)} = -\int_\Omega u^{(i)}(x)\partial_i\varphi^{(i)}(x)dx, \text{ for } i = 1, 2,$$

which yields

$$\lim_{n\to\infty} T_2^{(n)} = \sum_{i=1,2} \int_\Omega \varphi(x)\partial_i u^{(i)} dx.$$

Thanks to (27), we get:

$$\left|T_3^{(n)}\right| \le \lambda \mathrm{size}(\mathcal{D}^{(n)})\|p^{(n)}\|_{L^2(\Omega)}\|\tilde{\varphi}^{(n)}\|_{L^2(\Omega)} \le (\frac{\lambda}{\nu})^{\frac{1}{2}}\mathrm{diam}(\Omega)\|f\|_{(L^2(\Omega))^2}\|\varphi\|_{L^2(\Omega)}(\mathrm{size}(\mathcal{D}^{(n)}))^{1/2}.$$

Therefore, $\lim_{n\to\infty} T_3^{(n)} = 0$, and this in turn implies that:

$$\sum_{i=1,2} \int_\Omega \varphi(x)\partial_i u^{(i)}(x)dx = 0, \text{ for all } \varphi \in C_c^\infty(\Omega),$$

which proves that $u \in E(\Omega)$.

Proposition 7.2 [**Convergence properties of pressure terms**] Under Hypothesis (2), let $u \in H_0^1(\Omega) \cap H^2(\Omega)$ and let $\alpha > 0$ be given. Let $(\mathcal{D}^{(n)})_{n\in\mathbb{N}}$ be a sequence of admissible discretization of Ω in the sense of Definition 2.1, such that $\lim_{n\to\infty} \mathrm{size}(\mathcal{D}^{(n)}) = 0$ and $\mathrm{angle}(\mathcal{D}^{(n)}) \ge \alpha$, for all $n \in \mathbb{N}$. Let $C_{13} \in (0, +\infty)$, $\theta \in [0, 1)$ be given and let $p^{(n)} \in L_{\mathcal{D}^{(n)}}(\Omega)$ for all $n \in \mathbb{N}$, be a given sequence such that

$$\|p^{(n)}\|_{L^2(\Omega)} \le \frac{C_{13}}{\mathrm{size}(\mathcal{D}^{(n)})^\theta}, \ \forall n \in \mathbb{N}. \tag{34}$$

(Indeed, the existence of some data satisfying the above hypothesis is clear). Then

$$\lim_{n\to\infty}\left(\sum_{S\in\mathcal{V}^{(n)}}\sum_{K\in\mathcal{M}_S} u^{(i)}(x_K)A^{(i)}_{K,S}\, p^{(n)}_S - \int_\Omega \partial_i u^{(i)}(x)p^{(n)}(x)dx\right) = 0, \text{ for } i=1,2. \tag{35}$$

Proof Let $n\in\mathbb{N}$ and $i=1,2$ be given. Let us define

$$T_4^{(n,i)} = \sum_{S\in\mathcal{V}^{(n)}}\sum_{K\in\mathcal{M}_S} u^{(i)}(x_K)A^{(i)}_{K,S}\, p^{(n)}_S, \text{ and } T_5^{(n,i)} = \int_\Omega p^{(n)}(x)\partial_i u^{(i)}(x) \text{ for } i=1,2.$$

We then have

$$|T_4^{(n,i)} - T_5^{(n,i)}| \le \sum_{S\in\mathcal{V}^{(n)}} |p^{(n)}_S|\;|T^{(i)}_S|,$$

where, for $S\in\mathcal{V}^{(n)}$, the term $T^{(i)}_S$ is defined by

$$T^{(i)}_S = \sum_{K\in\mathcal{M}_S} A^{(i)}_{K,S}\, u^{(i)}(x_K) - \int_{\partial S} u^{(i)}(x)\mathbf{e_i}\cdot\mathbf{n}(x)d\gamma(x),$$

denoting by $\mathbf{e_i}$ the basis vector corresponding to the x coordinate. We apply Proposition 6.1 and we get

$$|T^{(i)}_S| \le C_6\,\mathrm{m}(S)|u^{(i)}|_{H^2(S)}.$$

The above inequality leads to

$$|T_4^{(n,i)} - T_5^{(n,i)}|^2 \le \mathrm{size}(\mathcal{D}^{(n)})^2 C_6{}^2 |u^{(i)}|^2_{H^2(\Omega)}\;\|p^{(n)}\|^2_{L^2(\Omega)}.$$

Thanks to Hypothesis (34) and to the Cauchy-Schwarz inequality, we get the existence of C_{14}, which depends on Ω and α but not on n such that

$$|T_4^{(n,i)} - T_5^{(n,i)}| \le C_6\,C_{13}\,|u^{(i)}|_{H^2(\Omega)}\mathrm{size}(\mathcal{D}^{(n)})^{1-\theta}.$$

The above inequality yields (35).

Proposition 7.3 **[Convergence properties for the divergence of the velocities]** Under Hypothesis (2), let $\alpha>0$ be given. Let $(\mathcal{D}^{(n)})_{n\in\mathbb{N}}$ be a sequence of admissible discretizations of Ω in the sense of Definition 2.1, such that $\lim_{n\to\infty}\mathrm{size}(\mathcal{D}^{(n)}) = 0$ and $\mathrm{angle}(\mathcal{D}^{(n)})\ge\alpha$, for all $n\in\mathbb{N}$. For all $n\in\mathbb{N}$, let $v^{(n)},\varphi^{(n)}\in H_{\mathcal{D}^{(n)}}(\Omega)$ be such that there exists $v\in L^2(\Omega)$ with $v^{(n)}\to v$ in $L^2(\Omega)$ as $n\to\infty$ and $\varphi\in H^1_0(\Omega)$ such that $\varphi^{(n)}\to\varphi$ in $L^2(\Omega)$ as $n\to\infty$. Let $C_{15}>0$ be given and for all $n\in\mathbb{N}$, let $\tilde\varphi^{(n)}\in L_{\mathcal{D}^{(n)}}(\Omega)$ be a sequence of functions such that,

$$\sum_{S\in\mathcal{V}^{(n)}}\sum_{K\in\mathcal{M}_S}(\tilde\varphi^{(n)}_S - \varphi^{(n)}_K)^2 \le C_{15}\,,\ \forall n\in\mathbb{N}. \tag{36}$$

(Indeed, the existence of such data is clear). Then

$$\lim_{n\to\infty}\sum_{S\in\mathcal{V}^{(n)}}\sum_{K\in\mathcal{M}_S} v^{(n)}_K A^{(i)}_{K,S}\tilde\varphi^{(n)}_S = -\int_\Omega v(x)\partial_i\varphi(x)dx, \text{ for } i=1,2. \tag{37}$$

Proof

Let $\varepsilon > 0$; let us first define $\bar{v} \in C_c^\infty(\Omega)$ such that $\|v - \bar{v}\|_{L^2(\Omega)} \leq \varepsilon$. For any $K \in \mathcal{M}^{(n)}$, we define $\bar{v}_K^{(n)} = \bar{v}(x_K)$, and we denote by $\bar{v}^{(n)} \in H_{\mathcal{D}^{(n)}}(\Omega)$ the piecewise function defined by $\bar{v}^{(n)}|_K = \bar{v}(x_K)$ for all $K \in \mathcal{D}^{(n)}$. It is obvious that $\bar{v}^{(n)}$ tends to $\bar{v}$ in $L^\infty(\Omega)$, as n tends to infinity. Let be $i = 1, 2$ given.

Thanks to the triangular inequality, we may write that,

$$\left| \int_\Omega v(x) \partial_i \varphi(x) dx + \sum_{S \in \mathcal{V}^{(n)}} \sum_{K \in \mathcal{M}_S} v_K^{(n)} A_{K,S}^{(i)} \tilde{\varphi}_S^{(n)} \right| \leq T_6 + T_7^{(n)} + T_8^{(n)} + T_9^{(n)}, \quad (38)$$

with

$$T_6 = \left| \int_\Omega v(x) \partial_i \varphi(x) dx - \int_\Omega \bar{v}(x) \partial_i \varphi(x) dx \right|,$$

$$T_7^{(n)} = \left| \int_\Omega \bar{\partial}_i v(x) (\varphi(x) - \tilde{\varphi}^{(n)}(x)) dx \right|,$$

$$T_8^{(n)} = \left| \int_\Omega \bar{\partial}_i v(x) \tilde{\varphi}^{(n)}(x) dx + \sum_{S \in \mathcal{V}^{(n)}} \sum_{K \in \mathcal{M}_S} A_{K,S}^{(i)} \bar{v}_K^{(n)} \tilde{\varphi}_S^{(n)} \right|,$$

$$T_9^{(n)} = \left| \sum_{S \in \mathcal{V}^{(n)}} \sum_{K \in \mathcal{M}_S} (v_K^{(n)} - \bar{v}_K^{(n)}) A_{K,S}^{(i)} \tilde{\varphi}_S^{(n)} \right|.$$

The choice of $\bar{v}$ immediately yields that

$$T_6 \leq \|\partial_i \varphi\|_{L^2(\Omega)} \|\bar{v} - v\|_{L^2(\Omega)} \leq \varepsilon \|\partial_i \varphi\|_{L^2(\Omega)}. \quad (39)$$

We now remark that

$$T_7^{(n)} \leq \|\bar{v}\|_{L^2(\Omega)} \|\varphi - \tilde{\varphi}^{(n)}\|_{L^2(\Omega)} \leq \|\bar{v}\|_{L^2(\Omega)} \|\varphi - \varphi^{(n)}\|_{L^2(\Omega)} + \|\bar{v}\|_{L^2(\Omega)} \|\varphi^{(n)} - \tilde{\varphi}^{(n)}\|_{L^2(\Omega)}.$$

Since we have

$$\|\varphi^{(n)} - \tilde{\varphi}^{(n)}\|_{L^2(\Omega)}^2 \leq \sum_{S \in \mathcal{V}^{(n)}} \sum_{K \in \mathcal{M}_S} \mathrm{m}(K \cap S) (\varphi_K^{(n)} - \tilde{\varphi}_S^{(n)})^2 \leq \pi \mathrm{size}(\mathcal{D}^n)^2 \sum_{S \in \mathcal{V}^{(n)}} \sum_{K \in \mathcal{M}_S} (\varphi_K^{(n)} - \tilde{\varphi}_S^{(n)})^2,$$

we deduce, thanks to Assumption (36), that

$$\|\varphi^{(n)} - \tilde{\varphi}^{(n)}\|_{L^2(\Omega)} \leq C_{15} \sqrt{\pi} \mathrm{size}(\mathcal{D}^n).$$

Hence, thanks to the fact that $\varphi^{(n)}$ tends to φ in $L^2(\Omega)$, we get that there exists $n_1 \in \mathbb{N}$ such that

$$T_7^{(n)} \leq \varepsilon \text{ for any } n \geq n_1. \quad (40)$$

Let us now turn to $T_8^{(n)}$. Since $(\varphi^{(n)})_{n \in \mathbb{N}}$ is a converging sequence in $L^2(\Omega)$, and thanks to (7), the sequence $(\tilde{\varphi}^{(n)})_{n \in \mathbb{N}}$ is also bounded in $L^2(\Omega)$. we may therefore apply Proposition 7.2 below to obtain that $T_8^{(n)} \to 0$ as $n \to +\infty$, and therefore, there exists $n_2 \in \mathbb{N}$ such that

$$T_8^{(n)} \leq \varepsilon \text{ for any } n \geq n_2. \tag{41}$$

Let us finally deal with $T_9^{(n)}$. Noting that, for any $K \in \mathcal{M}^{(n)}$, one has $\sum_{S\in\mathcal{V}_K^{(n)}} A_{K,S}^{(i)} = 0$, and that

$$\sum_{S\in\mathcal{V}^{(n)}} \sum_{K\in\mathcal{M}_S} (v_K^{(n)} - \bar{v}_K^{(n)}) A_{K,S}^{(i)} \tilde{\varphi}_S^{(n)} = \sum_{K\in\mathcal{M}^{(n)}} \sum_{S\in\mathcal{V}_K^{(n)}} (v_K^{(n)} - \bar{v}_K^{(n)}) A_{K,S}^{(i)} \tilde{\varphi}_S^{(n)},$$

one gets

$$T_9^{(n)} = |\sum_{K\in\mathcal{M}^{(n)}} \sum_{S\in\mathcal{V}_K^{(n)}} A_{K,S}^{(i)} (v_K^{(n)} - \bar{v}_K^{(n)})(\tilde{\varphi}_S^{(n)} - \varphi_K^{(n)})|.$$

Therefore, by the Cauchy-Schwarz inequality, and thanks to Assumption (36),

$$T_9^{(n)} \leq C_{15}{}^{\frac{1}{2}} \left(\sum_{K\in\mathcal{M}^{(n)}} (v_K^{(n)} - \bar{v}_K^{(n)})^2 \sum_{S\in\mathcal{V}_K^{(n)}} (A_{K,S}^{(i)})^2 \right)^{\frac{1}{2}}.$$

Now, thanks to Proposition 2.2,

$$\sum_{S\in\mathcal{V}_K^{(n)}} (A_{K,S}^{(i)})^2 \leq \sum_{S\in\mathcal{V}_K^{(n)}} \operatorname{diam}(K)^2 \leq C_2\, C_5\, \mathrm{m}(K).$$

Hence

$$T_9^{(n)} \leq (C_{15}\, C_2\, C_5\,)^{\frac{1}{2}} \|v^{(n)} - \bar{v}^{(n)}\|_{L^2(\Omega)}.$$

Since $v^{(n)}$ and $\bar{v}^{(n)}$ tend to v in $L^2(\Omega)$ as $n \to \infty$, we deduce from this last inequality that we may choose $n_3 \in \mathbb{N}$ such that, for all $n \geq n_3$,

$$T_9^{(n)} \leq \varepsilon. \tag{42}$$

From (38), (39), (40), (41) and (42), we obtain that

$$\lim_{n\to+\infty} \left| \sum_{S\in\mathcal{V}^{(n)}} \sum_{K\in\mathcal{M}_S} v_K^{(n)} A_{K,S}^{(i)} \tilde{\varphi}_S^{(n)} + \int_\Omega v(x) \partial_i \varphi(x) dx \right| = 0,$$

which concludes the proof.

8 Error estimate

We now prove the following error estimate.

Proposition 8.1 **[Error estimate for the penalized finite volume scheme in the linear case]** Under Hypotheses (2)-(4), let us suppose that there exists

$(\bar{u}, \bar{p}) \in (H_0^1(\Omega) \cap H^2(\Omega))^2 \times H^1(\Omega)$ solution of (1). Let $\mathcal{D}$ be an admissible discretization of Ω in the sense of Definition 2.1. Let $\lambda \in (0, +\infty)$ be given and $\alpha > 0$ with angle$(\mathcal{D}) \geq \alpha$.

Let $(u^{\mathcal{D}}, p^{\mathcal{D}}) \in (H_{\mathcal{D}}(\Omega))^2 \times L_{\mathcal{D}}(\Omega)$ be the solution to (23). We denote by $\bar{u}^{\mathcal{D}} \in (H_{\mathcal{D}}(\Omega))^2$ the piecewise constant function equal to $\bar{u}(x_K)$ on each control volume $K \in \mathcal{M}$. Then there exists $C_{16} > 0$, which only depends on Ω, ν and α, such that the following inequalities hold:

$$\|u^{\mathcal{D}} - \bar{u}^{\mathcal{D}}\|_{\mathcal{D}} \leq$$
$$C_{16}\,(h)^{1/4} \left(\|\bar{u}\|^2_{(H^2(\Omega))^2} + \|\bar{p}\|^2_{H^1(\Omega)}\right)^{1/2} \left(\sqrt{\lambda} + \frac{1}{\sqrt{\lambda}}\right)^{1/2}, \tag{43}$$

and

$$\|u^{\mathcal{D}} - \bar{u}\|_{(L^2(\Omega))^2} \leq$$
$$C_{16}\,(h)^{1/4} \left(\|\bar{u}\|^2_{(H^2(\Omega))^2} + \|\bar{p}\|^2_{H^1(\Omega)}\right)^{1/2} \left(\sqrt{\lambda} + \frac{1}{\sqrt{\lambda}}\right)^{1/2}. \tag{44}$$

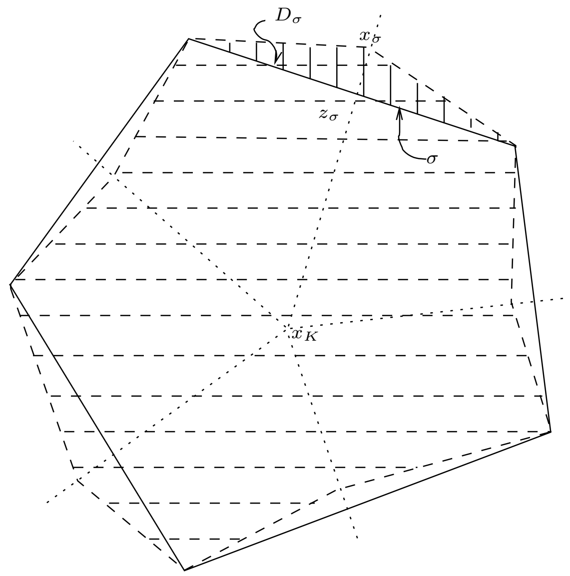


Figure 3: The domains D_σ (solid lines) and $\tilde{K}$ (dashed lines)

Proof In this proof, we denote by C_i, where i is an integer, real values which only depend on Ω, ν and α and not on h nor on λ. For all $\sigma \in \mathcal{E}$, let $\tilde{\sigma}$ be defined by $\tilde{\sigma} = [x_{S_1} x_\sigma] \cup [x_\sigma x_{S_2}]$, where x_{S_1} and x_{S_2} are the two vertices of σ. For all $K \in \mathcal{M}$, let $\tilde{K}$ be the polygonal subset of Ω whose edges are $\tilde{\sigma}$, for all $\sigma \in \mathcal{E}_K$ (see Figure (3)). We define $\varepsilon_{K,\sigma} = -1$ if $\tilde{\sigma} \subset \overline{K}$ else $\varepsilon_{K,\sigma} = 1$. Let D_σ be the triangle with edges σ and $\tilde{\sigma}$. Let us integrate the first equation of (1) on $\tilde{K}$. We get

$$\eta \int_K \bar{u}^{(i)}(x)dx - \nu \sum_{\sigma\in\mathcal{E}_K} \int_\sigma \nabla\bar{u}^{(i)}(x)\cdot \mathbf{n}_{K,\sigma} d\gamma(x) + \sum_{\sigma\in\mathcal{E}_K} \int_{\tilde{\sigma}} \bar{p}(x)\ \mathbf{e}_i \cdot \mathbf{n}_{\tilde{K}}(x) d\gamma(x) = \int_K f^{(i)}(x)dx + \sum_{\sigma\in\mathcal{E}_K} \varepsilon_{K,\sigma} \int_{D_\sigma} \left(f^{(i)}(x) + \nu\Delta\bar{u}^{(i)}(x) - \eta\bar{u}^{(i)}\right) dx,\ i=1,2, \tag{45}$$

where we denote by $\mathbf{n}_{\tilde{K}}(x)$ the unit vector outward to $\tilde{K}$ at point (x) and $\mathbf{e}_i$ the basis unit vector corresponding to the ith coordinate. Let us define

$$R^{(i)}_{\bar{u},K,\sigma} = \frac{1}{m_\sigma}\int_\sigma \nabla\bar{u}^{(i)}(x)\cdot\mathbf{n}_{K,\sigma} - \frac{1}{d_{K,\sigma}}(\bar{u}^{\mathcal{D},(i)}_\sigma - \bar{u}^{\mathcal{D},(i)}_K),\ i=1,2,$$

where $\bar{u}^{\mathcal{D},(i)}$ denotes the ith component of $\bar{u}^{\mathcal{D}}$, $\bar{u}^{\mathcal{D},(i)}_\sigma$ is defined by $\tau_{K,\sigma}(\bar{u}^{\mathcal{D},(i)}_\sigma - \bar{u}^{\mathcal{D},(i)}_K) + \tau_{L,\sigma}(\bar{u}^{\mathcal{D},(i)}_\sigma - \bar{u}^{\mathcal{D},(i)}_L) = 0$. It is then easy to see that $R^{(i)}_{\bar{u},K,\sigma} + R^{(i)}_{\bar{u},L,\sigma} = 0$ if $\sigma = K|L$, for $i=1,2$.

Let us also define

$$\tilde{R}^{(i)}_{\bar{u},K} = \bar{u}^{\mathcal{D},(i)}_K - \frac{1}{\mathrm{m}(K)}\int_K u^{(i)}(x)dx,\ i=1,2.$$

Thanks to the regularity assumption on the mesh angle$(\mathcal{D}) \ge \alpha$, we may apply Lemma 3.2 of [10], which states that there exists C_{17} depending only on $\bar{u}$ such that

$$|R^{(i)}_{\bar{u},K,\sigma}| + |\tilde{R}^{(i)}_{\bar{u},K}| \le C_{17}\,h,\ \text{ for } i=1,2. \tag{46}$$

With these definitions, we get from (45) that:

$$\eta\mathrm{m}(K)\bar{u}^{\mathcal{D}}_K - \nu\sum_{\sigma\in\mathcal{E}_K}\tau_{K,\sigma}(\bar{u}^{\mathcal{D}}_\sigma - \bar{u}^{\mathcal{D}}_K) + \sum_{\sigma\in\mathcal{E}_K}\int_{\tilde{\sigma}}\bar{p}(x)\ \mathbf{n}_{\tilde{K}}(x)d\gamma(x) = \int_K f(x)dx + \sum_{\sigma\in\mathcal{E}_K}\varepsilon_{K,\sigma}\int_{D_\sigma}(f(x)+\nu\Delta\bar{u}(x)-\eta\bar{u})\,dx + \nu\sum_{\sigma\in\mathcal{E}_K} m_\sigma R_{\bar{u},K,\sigma} + \mathrm{m}(K)\eta\tilde{R}_{\bar{u},K}. \tag{47}$$

Let us substract the above equation off the first equation of (24), and take the inner product with

$$\hat{u}_K = u_K - \bar{u}^{\mathcal{D}}_K,$$

and sum the resulting equations for $K\in\mathcal{M}$. Using (25), we get

$$\eta\|\hat{u}^{\mathcal{D}}\|^2_{L^2(\Omega)^2} + \nu\|\hat{u}^{\mathcal{D}}\|^2_{\mathcal{D}} + \lambda\ h\ \|p^{\mathcal{D}}\|^2_{L^2(\Omega)} = T_{10}+T_{11}+T_{12}+T_{13}, \tag{48}$$

with

$$T_{10} = \int_\Omega p^{\mathcal{D}}(x)\mathrm{div}_{\mathcal{D}}(\bar{u}^{\mathcal{D}})(x)dx,$$

$$T_{11} = \sum_{K\in\mathcal{M}}\sum_{\sigma\in\mathcal{E}_K}\hat{u}_K\cdot\int_{\tilde{\sigma}}\bar{p}(x)\ \mathbf{n}_{\tilde{K}}(x)d\gamma(x),$$

$$T_{12} = -\sum_{K\in\mathcal{M}}\sum_{\sigma\in\mathcal{E}_K} T_{14}^{(\bar u,K,\sigma)} \cdot \hat u_K,$$

and

$$T_{13} = -\sum_{K\in\mathcal{M}} \eta \mathrm{m}(K)\tilde R_{\bar u,K}\cdot \hat u_K,$$

where

$$T_{14}^{(\bar u,K,\sigma)} = \varepsilon_{K,\sigma}\int_{D_\sigma}(f(x)+\nu\Delta\bar u(x))dx + \nu m_\sigma R_{\bar u,K,\sigma}.$$

Let us first handle T_{10}. We have

$$T_{10} = \sum_{i=1,2}\sum_{S\in\mathcal{V}} p_S \sum_{K\in\mathcal{M}_S} A_{K,S}^{(i)}\bar u_K^{\mathcal{D},(i)}.$$

Thanks to $\mathrm{div}(\bar u)=0$, we may apply Proposition 6.1 and obtain:

$$\left|\sum_{i=1,2}\sum_{K\in\mathcal{M}_S} A_{K,S}^{(i)}\bar u_K^{\mathcal{D},(i)}\right| \le C_6\,\mathrm{m}(S)\|\bar u\|_{(H^2(S))^2}.$$

Thanks to the Cauchy-Schwarz inequality and to Estimate (27), we get

$$T_{10} \le C_{18}\,\frac{\mathrm{diam}(\Omega)\|f\|_{(L^2(\Omega))^2}h_{\mathcal{D}}}{(\nu\lambda h_{\mathcal{D}})^{1/2}}\|\bar u\|_{(H^2(\Omega))^2}.$$

Now remark that since $(\bar u,\bar p)$ is solution to (6), one has

$$\|f\|_{(L^2(\Omega))^2} \le C_{19}\,\|\bar u\|_{(H^2(\Omega))^2} + \|\bar p\|_{H^1(\Omega)}, \tag{49}$$

so that:

$$T_{10} \le C_{20}\left(\frac{h}{\lambda}\right)^{1/2}\left(\|\bar u\|^2_{(H^2(\Omega))^2} + \|\bar p\|^2_{H^1(\Omega)}\right). \tag{50}$$

We now turn to the study of T_{11}. Let us introduce the function $\bar p^{\mathcal{D}} \in L_{\mathcal{D}}(\Omega)$, defined by

$$\bar p^{\mathcal{D}}(x) = \bar p_S^{\mathcal{D}},\ \text{for a.e. } x\in S,\ \text{with } \bar p_S^{\mathcal{D}} = \frac{1}{\mathrm{m}(S)}\int_S \bar p(x)dx \text{ for all } S\in\mathcal{V}.$$

For an edge σ with vertices x_{S_1} and x_{S_2}, this function is then equal to the constant $\bar p_{S_1}^{\mathcal{D}}$ on $[x_{S_1}x_\sigma]$ and to the constant $\bar p_{S_2}^{\mathcal{D}}$ on $[x_\sigma x_{S_2}]$, thus it is integrable on $\tilde\sigma$. We then have $T_{11} = T_{15}+T_{16}$ with

$$T_{15} = \sum_{i=1,2}\sum_{K\in\mathcal{M}}\sum_{\sigma\in\mathcal{E}_K} \hat u_K^{(i)}\int_{\tilde\sigma}(\bar p-\bar p^{\mathcal{D}})(x)\ \mathbf{e}_i\cdot\mathbf{n}_{\tilde K}(x)d\gamma(x),$$

and

$$T_{16} = \sum_{i=1,2}\sum_{K\in\mathcal{M}}\sum_{\sigma\in\mathcal{E}_K}\left(\ \hat u_K^{(i)}\int_{\tilde\sigma}\bar p^{\mathcal{D}}(x)\ \mathbf{e}_i\cdot\mathbf{n}_{\tilde K}(x)d\gamma(x)\ \right).$$

By definition, $A^{(i)}_{K,S} = \int_{[x_{\sigma_1} x_S]\cup[x_S x_{\sigma_2}]} \mathbf{e}_i \cdot \mathbf{n}_{\tilde{K}}(x) d\gamma(x)$ if x_S is the common vertex to edges σ_1 and σ_2 in the trigonometric order; therefore, we have for all $K \in \mathcal{M}$:

$$\sum_{\sigma\in\mathcal{E}_K} \int_{\tilde{\sigma}} \bar{p}^{\mathcal{D}}(x)\ \mathbf{e}_i \cdot \mathbf{n}_{\tilde{K}}(x) d\gamma(x) = \sum_{S\in\mathcal{V}_K} A^{(i)}_{K,S}\ \bar{p}^{\mathcal{D}}_S,\ i = 1, 2.$$

Thus we get $T_{16} = T_{17} - T_{18}$ with

$$T_{17} = \sum_{S\in\mathcal{V}} \bar{p}^{\mathcal{D}}_S \sum_{i=1,2} \sum_{K\in\mathcal{M}_S} A^{(i)}_{K,S}\ u_K = \int_\Omega \bar{p}^{\mathcal{D}}(x) \mathrm{div}_{\mathcal{D}}(u)(x) dx,$$

and

$$T_{18} = \sum_{S\in\mathcal{V}} \bar{p}^{\mathcal{D}}_S \sum_{i=1,2} \sum_{K\in\mathcal{M}_S} A^{(i)}_{K,S}\ \bar{u}^{\mathcal{D},(i)}_K = \int_\Omega \bar{p}^{\mathcal{D}}(x) \mathrm{div}_{\mathcal{D}}(\bar{u}^{\mathcal{D}})(x) dx.$$

Thanks to the fact that $\|\bar{p}^{\mathcal{D}}\|_{L^2(\Omega)} \le \|\bar{p}\|_{L^2(\Omega)}$, we then have

$$|T_{17}| \le \|\bar{p}\|_{L^2(\Omega)} \|\mathrm{div}_{\mathcal{D}}(u)\|_{L^2(\Omega)}$$

which yields, thanks again to (27) and (49),

$$|T_{17}| \le C_{21}\,(\lambda\ h)^{1/2} \left(\|\bar{u}\|^2_{(H^2(\Omega))^2} + \|\bar{p}\|^2_{H^1(\Omega)} \right). \tag{51}$$

Using again Proposition 6.1, we have

$$|T_{18}| \le C_{22}\, h_{\mathcal{D}} \left(\|\bar{u}\|^2_{(H^2(\Omega))^2} + \|\bar{p}\|^2_{H^1(\Omega)} \right) \tag{52}$$

On the other hand, we have

$$T_{15} = \sum_{K\in\mathcal{M}} \sum_{i=1,2} \sum_{\sigma\in\mathcal{E}_K} (\hat{u}^{(i)}_K - \hat{u}^{(i)}_\sigma) \int_{\tilde{\sigma}} (\bar{p} - \bar{p}^{\mathcal{D}})(x)\ \mathbf{e}_i \cdot \mathbf{n}_{\tilde{K}}(x) d\gamma(x),$$

which yields, thanks to Young's inequality,

$$|T_{15}| \le \frac{\nu}{4} \|\hat{u}\|^2_{\mathcal{D}} + \frac{1}{\nu} \sum_{K\in\mathcal{M}} \sum_{\sigma\in\mathcal{E}_K} \frac{d_{K,\sigma}}{m_\sigma} \left(\int_{\tilde{\sigma}} (\bar{p} - \bar{p}^{\mathcal{D}})(x)\ \mathbf{e}_i \cdot \mathbf{n}_{\tilde{K}}(x) d\gamma(x) \right)^2.$$

Thanks to Proposition 6.2, we get, using $\tilde{\sigma} = [x_{S_1} x_\sigma] \cup [x_\sigma x_{S_2}]$ and thanks to the fact that $\mathbf{e_1} \cdot \mathbf{n}_{\tilde{K}}(x)$ is a constant smaller than 1 on $[x_{S_1} x_\sigma]$ and on $[x_\sigma x_{S_2}]$,

$$\left| \int_{\tilde{\sigma}} (\bar{p} - \bar{p}^{\mathcal{D}})(x)\ \mathbf{e}_i \cdot \mathbf{n}_{\tilde{K}}(x) d\gamma(x) \right| \le C_{10}\, h \left(|\bar{p}|_{H^1(S_1)} + |\bar{p}|_{H^1(S_2)} \right),\ i = 1, 2.$$

This leads to

$$|T_{15}| \le \quad \frac{\nu}{4} \|\hat{u}\|^2_{\mathcal{D}} + C_{23}\, h^2 \|\bar{p}\|^2_{H^1(\Omega)}, \tag{53}$$

thanks to (11) of Proposition 2.2.

We finally turn to the study of T_{12}. Using the fact that for $\sigma = K|L$, we have $T_{14}^{(i,\bar{u},K,\sigma)} + T_{14}^{(i,\bar{u},L,\sigma)} = 0$, we may write that

$$T_{12} = \sum_{i=1,2} \sum_{K\in\mathcal{M}} \sum_{\sigma\in\mathcal{E}_K} T_{14}^{(i,\bar{u},K,\sigma)} (\hat{u}_\sigma^{(i)} - \hat{u}_K^{(i)}),$$

We again apply Young's inequality. It yields

$$|T_{12}| \leq \frac{\nu}{4} \|\hat{u}\|_{\mathcal{D}}^2 + C_{24}\, h^2 \left(\|\bar{u}\|_{(H^2(\Omega))^2} + \|\bar{p}\|_{H^1(\Omega)}^2 \right), \tag{54}$$

thanks to easy bounds on $T_{14}^{(i,\bar{u},K,\sigma)}$ and to (46).

Let us then turn to T_{13}: Using Young's inequality, one gets that

$$T_{13} \leq \frac{1}{2} \sum_{i=1,2} \sum_{K\in\mathcal{M}} \eta \mathrm{m}(K) (\tilde{R}_{\bar{u},K}^{(i)})^2 + \frac{1}{2} \sum_{i=1,2} \sum_{K\in\mathcal{M}} \eta \mathrm{m}(K) (\hat{u}_K^{(i)})^2,$$

Using (46) and the discrete Poincaré inequality (see e.g. [6]) yields the existence of $C_{25} \in \mathbb{R}_+$, depending only on η, ν, $\bar{u}$ and Ω, such that

$$T_{13} \leq \frac{\nu}{4} \|\hat{u}\|_{\mathcal{D}}^2 + C_{25}\, h^2. \tag{55}$$

From (48)–(55), and using $2 \leq \sqrt{\lambda} + 1/\sqrt{\lambda}$ concludes the proof of (43). Thanks to the existence of C_{26} such that $\|\bar{u} - \bar{u}^{\mathcal{D}}\|_{(L^2(\Omega))^2} \leq C_{26}\, h \|\bar{u}\|_{(H^2(\Omega))^2}$, we get (44) using the discrete Poincaré inequality.

9 Numerical results

The implementation of the scheme was performed using the F90 language on a Unix system. The linear systems are solved using a direct method. The grid generator proceeds from a given number of initial grid blocks, which can be triangular or rectangular, describing the geometry, which are then uniformly refined as desired.

In the case of triangular meshes, we shall use two different sorts of refinement techniques in order to obtain rates of convergence. The first technique consists in starting from a given number of initial large triangles, describing the geometry, which are then uniformly refined as desired. We shall call this type of refinement the "homothetic" grids, since within a large triangle, all the refined triangles may be deduced from a large one by an homothety. Such a refinement procedure (and its pressure mesh counterpart) is illustrated on the two upper graphics of Figure 4. This type of refinement tends to yield some optimistic rates of convergence, because of the symmetries occuring between the small triangles, as we shall see in the sequel. Hence we designed another type of refinement, which consists in first cutting down the domain into as many smaller subdomains as wanted, and then meshing each subdomain with the triangular mesh. This procedure is depicted the two lower graphics of Figure 4, in the case where the domain Ω is a square, and one may see that even though there exists a mesh pattern, there is no symmetry between the small refined triangles.

Numerical experiments for the Stokes equations were performed on $\Omega = (0,1) \times (0,1)$, with various analytical solutions. Here we shall present the results for three cases:

- **Case 1:** $u^{(1)}(x) = 0$, $u^{(2)}(x) = x^{(1)}(1 - x^{(1)})$, $p(x) = 0$. Note that in this case, the velocity is a polynomial function of degree 2, and therefore the consistency error on the diffusion flux is zero.

- **Case 2:** The velocities are $u^{(1)}(x) = -\partial_2 \Phi(x)$, and $u^{(2)}(x) = \partial_1 \Phi(x)$, where $\Phi(x) = 1000[x^{(1)}(1 - x^{(1)})x^{(2)}(1 - x^{(2)})]^2$, the pressure being given by $p(x) = (x^{(1)})^2 + (x^{(2)})^2 - 2/3$, for all $x \in \Omega$.

- **Case 3:** The velocities are $u^{(1)}(x) = \frac{1}{2}\sin(2\pi x^{(1)})\cos(2\pi x^{(2)})$, $u^{(2)}(x) = -\frac{1}{2}\cos(2\pi x^{(1)})\sin(2\pi x^{(2)})$, and the pressure $p(x) = \frac{1}{8}\cos(4\pi x^{(1)})\sin(4\pi x^{(2)})$.

In all cases, the right hand side f is then computed with (1) for the Stokes equations, taking $\nu = 1$ and $\eta = 0$.

The finite volume scheme (24)–(25) (which we call "present scheme" here) for which we proved convergence in the above sections was tried against the former scheme which was studied in [6], [1] and [3], and which we shall refer to as "VF-P1 scheme". Recall that these two schemes coincide on equilateral meshes.

The rate of convergence is obtained by computing $\ln \|\psi_h - \psi_{ex}\|$, where ψ_h represents the computed approximation of the exact quantity ψ_{ex} (component of the velocity or pressure) and identifying it as an affine function of the step size h: $\ln \|\psi_h - \psi_{ex}\| = \rho h + C$; when h is small enough, ρ is a good candidate for the convergence rate.

The rate of convergence for the three cases for both schemes on different meshes are given in Table 1. The rates of convergence are identical for both components of the velocities in all three cases. In the case of the former scheme, we have only given the results for triangular meshes, since there is no sound basis for a definition of this scheme on rectangular meshes (see [6]); in fact, this scheme implemented on rectangles does not converge. Note also that for the solution of case 1, we do not give the rate of convergence of the present scheme on rectangles, because the precision obtained is the machine precision for any mesh. Indeed, this is due to the fact that the solution is a second order polynomial, and therefore the consistency error on the flux is zero. Hence we obtain the exact solution (up to machine precision) for this case. One may also see that on rectangles, the order of convergence is equal to 2 for both velocities and pressure, for case 2; however, for case 3, it is still close to 1 for the velocities but drops down below 1 for the pressure.

One may see that the rate of convergence of the present scheme is always higher than that of the former scheme. The difference is particularly drastic for Case 2 on the non homogeneous mesh, since the rate of convergence of the present scheme is 1.66 for the velocities, and .52 for the pressure, while the VF-P1 scheme does not seem to really converge. Of course, as expected, the rates of convergence are better on the homothetic meshes than on the non-homothetic ones, and this is particularly true for the VF-P1 scheme; indeed, this illustrates the fact that the VF-P1 scheme was shown to converge on equilateral meshes, since the homothetic meshes are "closer" to the equilateral meshes than the non-homothetic ones.

	VF-P1 scheme		Present scheme	
	hom.	non-hom.	hom.	non-hom
u	1.91	1.03	1.91	1.80
p	1.00	0.10	1.00	.95

Case 1: $u^{(1)}(x) = 0$, $u^{(2)}(x) = x^{(1)}(1 - x^{(1)})$, $p(x) = 0$.

	VF-P1 scheme		Present scheme		
	hom.	non-hom.	hom.	non-hom	rectangles
u	1.95	0.05	1.99	1.66	1.99
p	1.03	-10^{-5}	1.07	0.52	1.51

Case 2 $u^{(1)}(x) = -\partial_2\Phi(x)$, $u^{(2)}(x) = \partial_1\Phi(x)$,
$\Phi(x) = [x^{(1)}(1 - x^{(1)})x^{(2)}(1 - x^{(2)})]^2$, $p(x) = (x^{(1)})^2 + (x^{(2)})^2$.

	VF-P1 scheme		Present scheme		
	hom.	non-hom.	hom.	non-hom	rectangles
u	1.93	0.25	2.00	2.00	2.00
p	1.00	0.05	1.50	1.12	2.00

Case 3 $u^{(1)}(x) = \frac{1}{2}\sin(2\pi x^{(1)})\cos(2\pi x^{(2)})$, $u^{(2)}(x) = -\frac{1}{2}\cos(2\pi x^{(1)})\sin(2\pi x^{(2)})$,
$p(x) = \frac{1}{8}\cos(4\pi x^{(1)})\sin(4\pi x^{(2)})$.

Table 1: Rates of convergence of both schemes on the different meshes and for the three cases of analytical solutions

In figure 5, we show the location of the error in velocity and pressure for the velocity and the pressure for both the VF-P1 scheme and the present one, on a homothetic mesh. It is clear that the error for the VF-P1 scheme is much higher at the interface between the large triangles of the original mesh, where no symmetry is there to help out. The scheme studied here is far less sensitive to these interfaces.

10 Concluding remarks

In this paper we introduced a finite volume scheme for the solution of the stationary Stokes equations on general meshes in two dimensions. We prove an error estimate for a penalized version of the scheme.

From the numerical results, we notice that the scheme which was introduced in [6] and proved to converge for equilateral meshes in [6] and [3] requires the modification made here in order to converge on general triangular meshes, contrarily to the expectations of [1].

The numerical results show that the present theoretical error estimate is not sharp. In order to improve this error estimate, an analytical estimate on the discrete pressure should be obtained. This has been done in the case of the MAC scheme (velocities at the edges of the mesh, [16]) where a $L^2(\Omega)$-estimate on the discrete pressures can be obtained [23], [2], thanks to a De Rham theorem, connecting the components of the gradient of the pressure with the Laplacian of functions of $H_0^1(\Omega)$ (see [17]). The use of the De Rham theorem is unfortunately not straightforward in our case, due mainly to the L^2 regularity of the discrete pressure together with consistency properties.

From the numerical results, we also show that the scheme which was introduced in [6] and proved to converge for equilateral meshes in [6] and [3] requires the modification made here in order to converge on general triangular meshes, contrarily to the expectations of [1].

Nevertheless, the observed accuracy of our numerical results indicates that this method is convenient for the calculation of an incompressible flow on triangular meshes. We do need to mention some oscillations of the pressure in one case. Furthermore, thanks to the location of the discrete unknowns, the finite volume code can be easily coupled with thermal or chemical codes if needed.

The scheme has also been developed for the nonlinear case, that is the Navier-Stokes equations, and yields interesting results on wellknown benchmarks such as the backward step.

References

[1] A. Alami-Idrissi A., M. Atounti, An error estimate for finite volume methods for the Stokes equations, Journal of Inequalities in Pure and Applied Mathematics, Volume 3, Issue 1, Article 17, 2002.

[2] Ph. Blanc, Schémas volumes finis pour la résolution des équations de Stokes", Ph.D. thesis, in preparation, Université de Marseille.

[3] Ph. Blanc, R. Eymard and R. Herbin, An error estimate for finite volume methods for the Stokes equations on equilateral triangular meshes, Ph. Blanc, R. Eymard and R. Herbin, Num. Meth. PDE, Published Online: 8 Apr 2004

[4] S. Boivin, F. Cayré, J.M. Hérard, A finite volume method to solve the Navier-Stokes equations for incompressible flows on unstructured meshes, Int. J. Therm. Sci., 38, 806-825, 2000.

[5] Y. Coudière, T. Gallouët and R. Herbin, Discrete Sobolev Inequalities and L^p Error Estimates for Finite Volume Solutions of Convection Diffusion Equations, M2AN, 35,4, 767-778, 2001.

[6] R. Eymard, T. Gallouët and R. Herbin, Finite Volume Methods, Handbook of Numerical Analysis, Vol. VII, pp. 713-1020. Edited by P.G. Ciarlet and J.L. Lions (North Holland),2000.

[7] R. Eymard and R. Herbin, A cell-centered finite volume scheme on general meshes for the Stokes equations in two dimensions, CRAS, Mathématiques. 125–128, t.337, 2, 2003.

[8] R. Eymard and R. Herbin, A finite volume scheme on general meshes for the steady Navier-Stokes equations in two space dimensions, to appear in the Proceedings of ENUMATH03, Prague, August 2003.

[9] R. Eymard, T. Gallouët and R. Herbin, Convergence of finite volume approximations to the solutions of semilinear convection diffusion reaction equations, Numer. Math., 82, 91-116, 1999.

[10] T. Gallouët, R. Herbin and M.H. Vignal, Error estimates for the approximate finite volume solution of convection diffusion equations with general boundary conditions, SIAM J. Numer. Anal, 37, 6, 1935 - 1972, 2000.

[11] V. Girault, P.-A. Raviart, Finite element methods for the Navier-Stokes equations: Theory and algorithms, Springer, Berlin, 1986.

[12] R. Glowinski, Numerical Methods for fluids (Part3), Handbook of Numerical Analysis, Vol. IX. Edited by P.G. Ciarlet and J.L. Lions (North Holland), 2003.

[13] M.D. Gunzburger, Finite element methods for viscous incompressible flows, A guide to thoery, practice, and algorithms, Computer Science qnd Scientific Computing, Academic Press 1989.

[14] M.D. Gunzburger and R.A Nicolaïdes Incompressible computational fluid dynamics, Cambridge University Press, 1993.

[15] J.L. Lions, Quelques méthodes de résolution des problèmse aux limites non linéaires, Dunod, Paris, 1969

[16] F.H. Harlow, J.E. Welch, Numerical calculation of time dependent viscous incompressible flow of fluids with free surface, Phys. Fluids, 8, 2182-2189, 1965.

[17] J. Nečas, Equations aux dérivées partielles, Presses de l'Université de Montréal, 1965.

[18] J.C. Nedelec, Notions sur les techniques d'éléments finis Mathématiques et applications Ellipses, 1980.

[19] R.A. Nicolaïdes, Analysis and Convergence of the MAC Scheme I. The Linear Problem. *SIAM J. Numer. Anal.*, Vol. 29, p. 1579–1591, 1992.

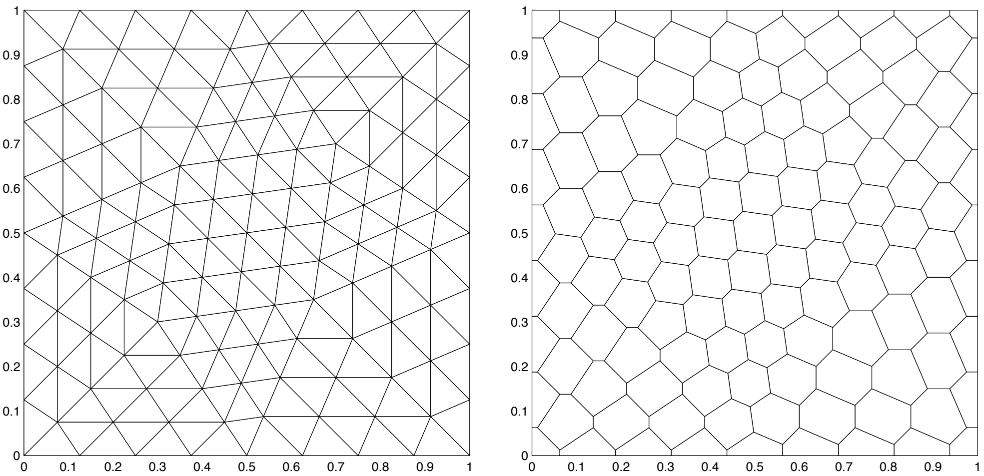
[20] S.V. Patankar, Numerical Heat Transfer and Fluid Flow, Series in Computational Methods in Mechanics and Thermal Sciences, Minkowycz and Sparrow Eds. (Mc Graw Hill), 1980.

[21] R. Peyret and T. Taylor, Computational methods for fluid flow, Springer, New-York, 1993.

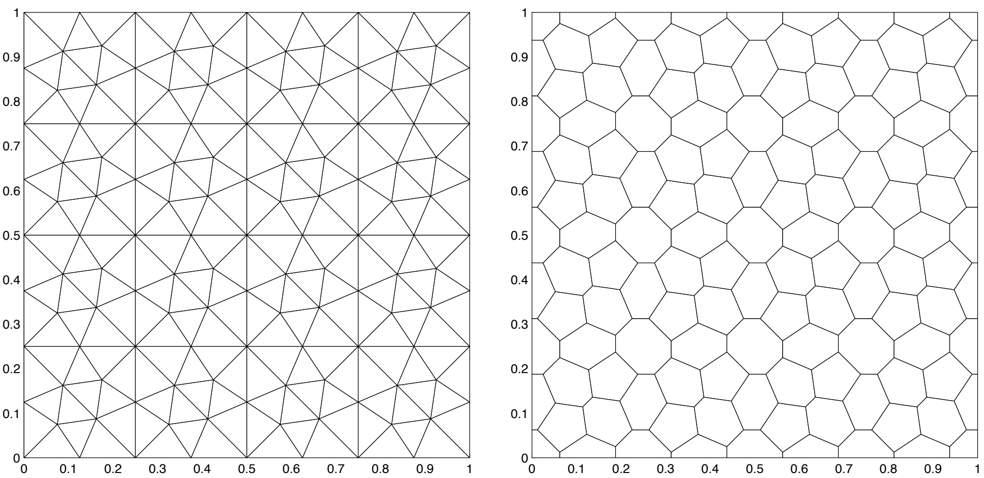
[22] O. Pironneau, Finite element methods for fluids, John Wiley and sons, 1989.

[23] D. Shin, J.C. Strikwerda, Inf-sup Conditions for Finite Difference Approximations of the Stokes Equations, J. Austral. Math. Soc. Ser. B, Vol. 39, p. 121–134, 1997.

[24] R. Temam, Navier-Stokes Equations, Studies in mathematics and its applications, J.L. Lions, G. Papanicolaou, R.T. Rockafellar Editors, North-Holland, 1977.

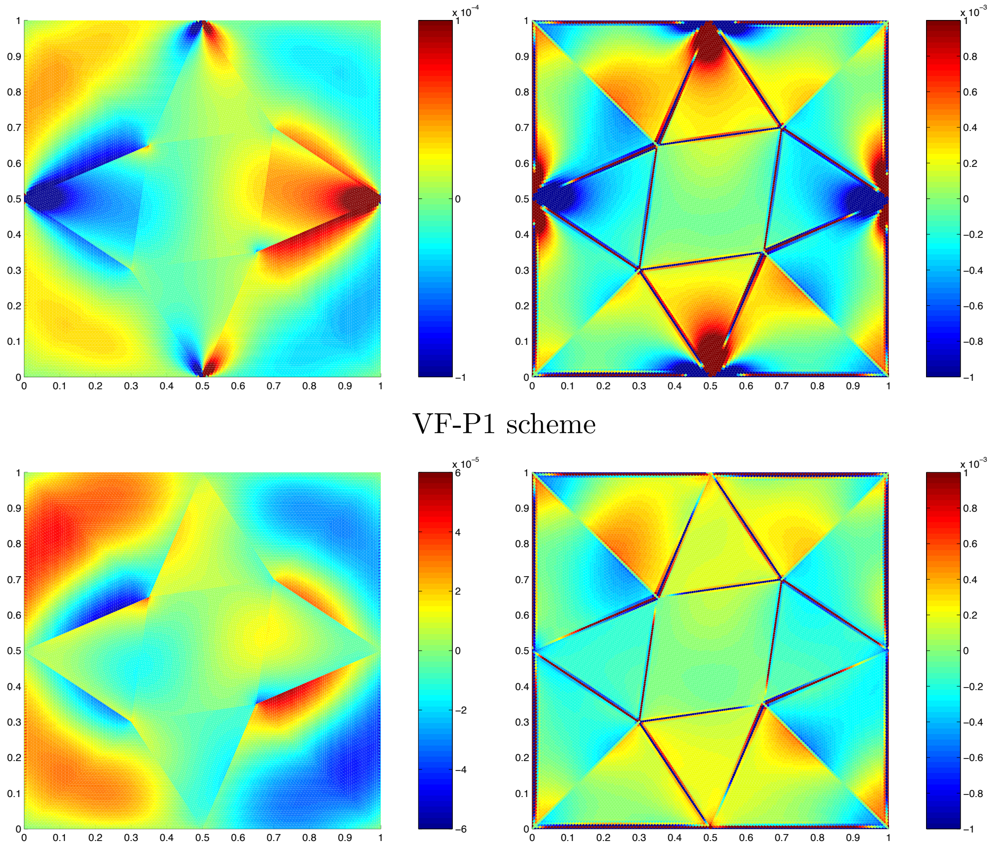


Homothetic refinement



Non-homothetic refinement

Figure 4: Velocity and pressure meshes for two types of refinement techniques



VF-P1 scheme

Present scheme

Figure 5: Relative error on velocity and pressure for both schemes on homothetic mesh